Triple Product L-function

Dmitriy Shvydkoy
Faculty Mentor: Michael Woodbury

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The representation theory of groups is a key part and will be used extensively throughout the project.

Gelfand Pairs and Triple Products

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Fact: When $F = \mathbb{F}_q$ it is not a Gelfand pair, as there exists an irreducible representation (ρ, V) of G such that $\{v \in V \mid \rho(h)(v) = v \ \ \forall h \in H\}$ has dimension two.

Triple Product L-function

Given modular forms f, g, h, define L-functions:

$$L(s, f) = \sum_{n \geq 1} \frac{a_n}{n^s}, \quad L(s, g) = \sum_{n \geq 1} \frac{b_n}{n^s}, \quad L(s, h) = \sum_{n \geq 1} \frac{c_n}{n^s}.$$

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The analytic properties (i.e. poles and zeros) of this function are of deep interest in number theory and physics.

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$$
\begin{aligned}
&= \frac{(\chi_3\chi_5)(-1)}{q^2(q^2-1)} \sum_{r\in\mathbb{F}_q^\times} \sum_{a\in\mathbb{F}_q^\times} (\chi_1\chi_3\chi_5)(a)(\chi_3^{-1}\chi_4\chi_5^{-1}\chi_6)(r) \\
&\quad + \frac{(\chi_3\chi_5)(-1)}{q^2(q^2-1)} \sum_{r\in\mathbb{F}_q^\times} \sum_{\substack{s\in\mathbb{F}_q^\times \\ s\neq r}} \sum_{a\in\mathbb{F}_q^\times} (\chi_1\chi_3\chi_5)(a)\psi(\tfrac{a}{r} - \tfrac{a}{s})(\chi_3^{-1}\chi_4)(r)\overline{\psi}(r-s)(\chi_5^{-1}\chi_6)(s)
\end{aligned}
$$