Triple Product L-function

Dmitriy Shvydkoy
Faculty Mentor: Michael Woodbury
Graduate Mentor: Dennis Hou

DIMACS REU

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Previous Group

NOTES FROM TRIPLE PRODUCT L-FUNCTION REU AT COLUMBIA UNIVERSITY

ALAN DU, KEVIN KWAN, LORI LEU, MICHAEL WOODBURY, ANTON WU

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- $f(\frac{az+b}{cz+d}) = (cz+d)^k f(z)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), z \in \mathbb{H}$.
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This imples f has the Fourier expansion $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$.

L-functions

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Dirichlet's Theorem: For any coprime integers a and d, there are infinitely many primes in the sequence

$$a, a+d, a+2d, a+3d, ...$$

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Ichino's Theorem:

$$\left| \int_{SL_2(\mathbb{Z}) \backslash \mathbb{H}} \phi_1 \phi_2 \phi_3 \right|^2 \doteq L\left(\frac{1}{2}, \phi_1 \otimes \phi_2 \otimes \phi_3\right) \prod_{p \leq \infty} I_p(\phi_p),$$

where $I_p(\phi_p)$ is a local trilinear form.

Group Representations

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The representation theory of groups is a key part and is used extensively throughout the project.

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Define the **Borel subgroup** of $GL_2(\mathbb{F}_q)$ as the following:

$$B = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \mid \alpha, \delta \in \mathbb{F}_q^*, \beta \in \mathbb{F}_q \right\}$$

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There is a standard way to extend a representation from a subgroup to the whole group called **induction**.

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- Cuspidal representations of dimension $q - 1$

Trilinear Forms

For some choice of irreducible representations π_1, π_2, π_3 of $GL_2(\mathbb{F}_q)$, and some choice of vectors $f_1 \in V_{\pi_1}$, $f_2 \in V_{\pi_2}$, $f_3 \in V_{\pi_3}$, define the **matrix coefficient** by

$$\Phi(g) = \langle \pi_1(g) f_1, \bar{f}_1 \rangle \langle \pi_2(g) f_2, \bar{f}_2 \rangle \langle \pi_3(g) f_3, \bar{f}_3 \rangle$$

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Then the **trilinear form** is

$$\frac{1}{|GL_2(\mathbb{F}_q)|} \sum_{g \in GL_2(\mathbb{F}_q)} \Phi(g)$$

Progress

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- G-invariant inner product, so satisfies

$$\langle v, w\rangle = \langle \pi(g)v, \pi(g)w\rangle \quad v, w \in V_\pi, g \in G$$

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- Function that computes cuspidal representations
  - First attempted to use Whittaker model
  - Then implemented using the direct construction of the cuspidal representations
- Working with a student from the previous group, we used this code to compute trilinear forms on a specific combination of irreducible representations and formed a conjecture which is now proved.

Future Work

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- Prove these conjectures.

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https://github.com/Dmitriy435/DIMACS