Graph Pebbling on Graph Product

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Introduction to Pebbling

**Definition**

A *configuration* of pebbles is a distribution of pebbles onto the vertices of a graph $G$.

**Definition**

A *pebbling move* consists of removing two pebbles at a vertex and placing one pebble at an adjacent vertex.
A configuration is *solvable* if every vertex could be reached by a pebble after a series of pebbling moves.
Definition

The *pebbling number* of a graph, denoted $\pi(G)$, is the number of pebbles needed to guarantee that all configurations of $\pi(G)$ pebbles are solvable. We call a graph *class-0* if $|V(G)| = \pi(G)$.

Goal of Project: Determine or set bounds on the pebbling number of certain graphs.
The Cartesian product $G \square H$ is the graph given by

$$V(G \square H) = \{(g, h) : g \in V(G), h \in V(H)\}$$

and

$$E(G \square H) = \{\{(g_1, h_1), (g_2, h_2)\} : [g_1 = g_2 \text{ and } \{h_1, h_2\} \in E(H)] \text{ or } [\{g_1, g_2\} \in E(G) \text{ and } h_1 = h_2]\}$$
Example: $P_2 \Box P_3$
Example: $K_5 \Box K_5$
**Conjecture (Graham)**

\[ \pi(G \Box H) \leq \pi(G)\pi(H) \]

**Known for some cases:**
- tree with a tree
- cycle with a cycle
- products of paths

**Implied by Graham’s:** If \( G \) is class-0, then \( G \Box G \) is class-0.
What kind of Graph Product is class-0

Can we have a graph $G$ that is not class-0, but the Cartesian product $G \square G$ is class-0?

A possible answer

The family of graphs with diameter 2 and 2-connectivity but not class-0.

Figure: The simplest graph of the family
An Alternate Problem: Graham’s Conjecture

Does the inequality $\pi(G \Box G) \leq \pi(G) \times \pi(G)$ hold for this graph?

Figure: $\pi(G) = 7, \pi(G) \times \pi(G) = 49$
An Alternate Problem

Can we pebble every vertex on this graph with any distribution of 49 pebbles on the vertices?
A class-0 subgraph within the product graph

A simplified $K_3 \square K_3$
A class-0 subgraph within the product graph

A simplified $K_3 \square K_3$
A class-0 subgraph within the product graph

A simplified $K_3 \square K_3$

The other vertices have less than 9 pebbles
Select the target vertex
The red vertices are in the same $K_3 \square K_3$

The sum of the pebbles on the red vertices is no more than 24. The sum of the pebbles on the rest of the 11 vertices is at least 12.
Consider the following case

Each of the red vertices has exactly one pebble, and the sum of their pebbles reaches the maximum 24. Since there are 12 pebbles left on the remaining 11 vertices, by Pigeonhole Principle we can move at least one pebble to any of the $K_3 \Box K_3$, in which we have enough pebbles to reach the target vertices.
More cases and details to work on

Three kinds of vertices

- Vertex with degree 4. Each in 1 copy of $K_3 \square K_3$.
- Vertex with degree 6. Each in 3 copies of $K_3 \square K_3$.
- Vertex with degree 8. Each in 9 copies of $K_3 \square K_3$. 
Open Problems Following

Generalize

- Can we show that other graphs of diameter 2 and 2-connectivity also holds for Graham's conjecture?
- Can we show that the all the products of such graphs are class-0?
- Can we construct similar graphs with similar symmetrical structure and complete subgraphs?
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