DIMACS REU 2018
Project: Sphere Packings and Number Theory

Zachary Stier
Mentor: Prof. Alex Kontorovich

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Apollonian circle packings

What is an Apollonian circle packing?

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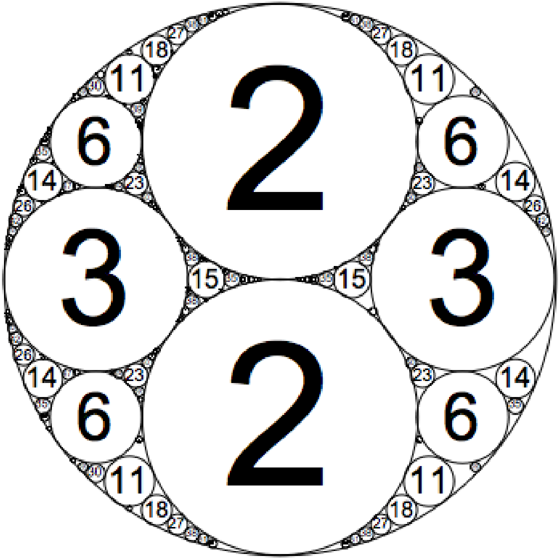
Here is an illustrative example:



27
18
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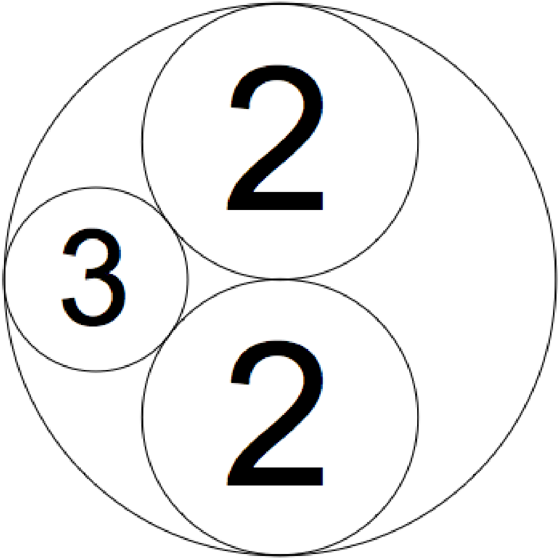
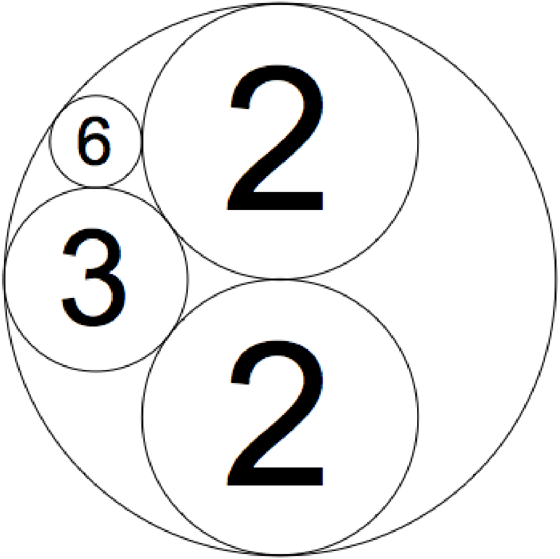
Apollonian circle packings

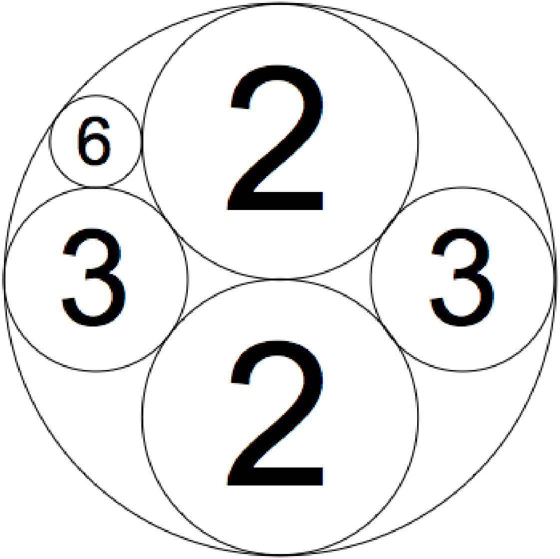
Here's how this is made:



Starting configuration



Add another circle



Continue adding circles...

Apollonian circle packings

…and eventually we end up with a *space-filling* circle packing, in that every point on the interior of the bounding circle belongs to a circle or lies on a boundary.

Apollonian circle packings

So, what else can we do with Apollonian circle packings? Are there other examples? Can we go into higher dimensions?

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Answers: Yes and sort-of.

Other examples in the plane

Given any initial configuration, there is a unique packing. Further, given an initial configuration of four circles with integer bends, the induced packing is guaranteed to have entirely integer bends.

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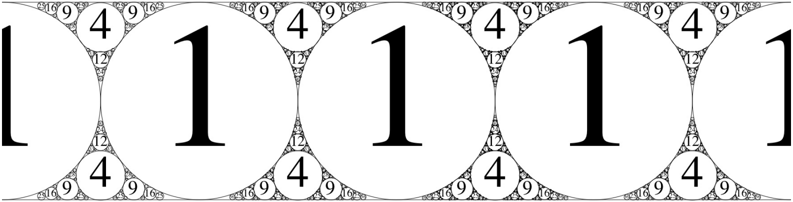
Given any initial configuration, there is a unique packing. Further, given an initial configuration of four circles with integer bends, the induced packing is guaranteed to have entirely integer bends. This is a consequence of Descartes' kissing circles theorem,

$$\sum_{i=1}^{4} b_i^2 = \frac{1}{2}\left(\sum_{i=1}^{4} b_i\right)^2 .$$



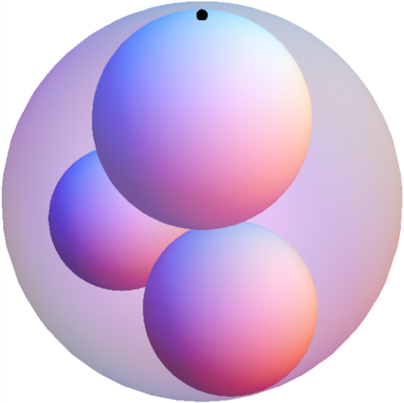
A *strip packing*. Lines are "circles of radius ∞" and thus have bend 0.
Image source: Prof. Kontorovich

Higher dimensions?

Descartes' theorem extends to dimension 3: starting with any integer-bend spheres that are mutually tangent,

$$\sum_{i=1}^{5} b_i^2 = \frac{1}{3}\left(\sum_{i=1}^{5} b_i\right)^2 .$$

This lets us consider configurations such as…



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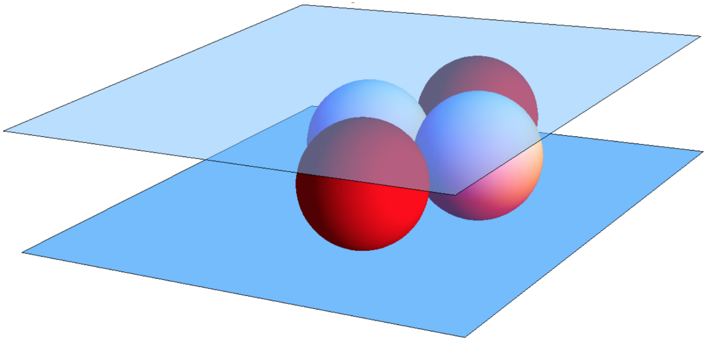

Another strip packing. Again, planes are “spheres of radius ∞” and thus have bend 0.
Image source: Prof. Kontorovich

Higher dimensions?

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Higher dimensions???

In 4 dimensions and above, we can't just work with *any* initial configuration, because it turns out that it's possible for the spheres dictated by Descartes' theorem to start intersecting each other.

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In 4 dimensions and above, we can't just work with *any* initial configuration, because it turns out that it's possible for the spheres dictated by Descartes' theorem to start intersecting each other. We therefore need to look at nice classes of circle packings in these higher dimensions.

Motivation for "nice" packings

A very nice property of the Apollonian packings in the plane is that they can be modeled as iterated *reflections* on the initial configuration.

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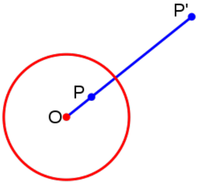
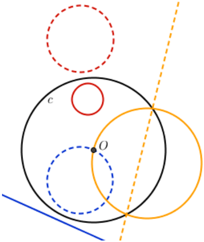
By reflections, we refer to a generalized notion of mirrors:



The inverse of a point.
Image source: Wikipedia



The inverses of more objects.
Image source: Malin Christersson

and we see that reflection about a line is equivalent to inversion about a really big (i.e. “radius ∞”) circle.

Motivation for “nice” packings

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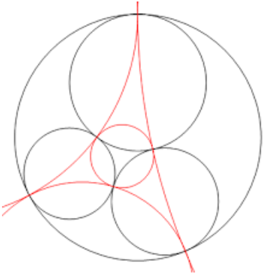

Image source: Arseniy Sheydvasser

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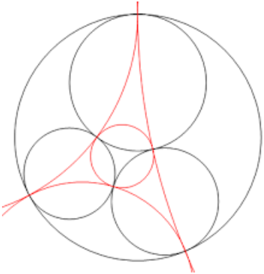

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Because reflection about a circle is its own inverse, we see that if we keep applying the red circles to the black ones, the action of each red one leaves the resultant figure of black circles unchanged.

Crystallographic packings

This is the inspiration for Kontorovich and Nakamura's concept of a *crystallographic packing*: a sphere packing (not necessarily in only 2 dimensions) for which the group of symmetries is generated by reflections.

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(In this context, the "group of symmetries" refers to the set of functions on the space that preserve the structure of the packing.)

Further restrictions

Crystallographic packings give us a lot to work with, but we can't increase the dimension forever.

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My project

Broadly speaking, I will be investigating these sphere packings in higher dimensions.

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Broadly speaking, I will be investigating these sphere packings in higher dimensions. To start, I will be working on the classification and cataloguing of the known crystallographic packings. Later in the program I aim to delve deeper and further investigate these higher-dimensional packings.

Acknowledgements

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