Apollonian Circle Packing

This is an Apollonian circle packing:
Apollonian Circle Packing

Here’s how we construct it:

- Start with three mutually tangent circles
Apollonian Circle Packing

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Apollonian Circle Packing

Here’s how we construct it:

- Start with three mutually tangent circles
- Draw two more circles, each of which is tangent to the original three
Apollonian Circle Packing

- Start with three mutually tangent circles
- Draw two more circles, each of which is tangent to the original three
Apollonian Circle Packing

- Start with three mutually tangent circles
- Draw two more circles, each of which is tangent to the original three
- Continue drawing tangent circles, densely filling space
Apollonian Circle Packing

These two images actually represent the same circle packing! We can go from one realization to the other using circle inversions.
Apollonian Circle Packing

These two images actually represent the same circle packing! We can go from one realization to the other using circle inversions.
Circle Inversions

Circle inversion sends points at a distance of $rd$ from the center of the mirror circle to a distance of $r/d$ from the center of the mirror circle.
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Circle Inversions

Circle inversion sends points at a distance of $rd$ from the center of the mirror circle to a distance of $r/d$ from the center of the mirror circle.

- We apply circle inversions to both spheres and planes, where planes are considered spheres of infinite radius.
- Circle inversions preserve tangencies and angles.

Source: Malin Christersson
Apollonian Circle Packing
Apollonian Circle Packing

To generate a packing, invert the blue line about the reds
Apollonian Circle Packing

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Apollonian Circle Packing

To generate a packing, invert the blue line about the reds
Sphere Packings: Definition

The sphere packings we’ve examined this summer are configurations where the spheres:

- have varying radii
- are oriented to have mutually disjoint interiors
- densely fill up space
Hyperbolic Geometries

- There is a surprising connection between sphere packings and non-Euclidean geometries.
Hyperbolic Geometries

- There is a surprising connection between sphere packings and non-Euclidean geometries.

- Euclidean geometry is characterized by Euclid’s *parallel postulate*, which states that the angles formed by two lines intersecting on one side of a third line sum to be less than \( \pi \) radians.

These geometries have several models which are each used as is necessary.

For now, we are going to focus on the upper half-space model of $\mathbb{H}^{n+1}$: consider $\mathbb{R}^{n+1}$, subject to $x_0 > 0$. This space has its own metric, and has as its boundary $\mathbb{R}^n$. 
Hyperbolic Geometries

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- For now, we are going to focus on the upper half-space model of $\mathbb{H}^{n+1}$: consider $\mathbb{R}^{n+1}$, subject to $x_0 > 0$. This space has its own metric, and has as its boundary $\mathbb{R}^n$.

- Because of the different metric, planes in $\mathbb{H}^{n+1}$ are actually hemispheres, with their circumferences lying in $\mathbb{R}^n$ (i.e., the subset $x_0 = 0$).

- Conveniently, we’ve already been looking at spheres lying in $\mathbb{R}^n$! We can “continue our configurations upwards” in what is known as the Poincaré extension.
Poincaré Extension
Hyperbolic Geometries

Another useful model of hyperbolic space is the two-sheeted hyperboloid model.
Another useful model of hyperbolic space is the **two-sheeted hyperboloid** model.

- A quadratic form $Q$ is a polynomial where each term is of degree exactly 2. It can be used to define an inner product space.

- We’re working on the top sheet of this 2-sheeted hyperboloid model of hyperbolic space, where all vectors $v$ satisfy $\langle v, v \rangle_Q = -1$
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- We’re working on the top sheet of this 2-sheeted hyperboloid model of hyperbolic space, where all vectors $v$ satisfy

  $$\langle v, v \rangle_Q = -1$$

Where did this quadratic form $Q = -1$ come from? Circle inversions!
From Circle Inversions to Quadratic Forms
From Circle Inversions to Quadratic Forms
From Circle Inversions to Quadratic Forms
From Circle Inversions to Quadratic Forms
From Circle Inversions to Quadratic Forms
From Circle Inversions to Quadratic Forms

Inverted circle’s diameter:

\[
\frac{1}{\|z\|-r} - \frac{1}{\|z\|+r}
\]
From Circle Inversions to Quadratic Forms

\[
\hat{d} = \frac{1}{|z| - r} - \frac{1}{|z| + r}
\]

\[
\hat{d} = \frac{2r}{|z|^2 - r^2}
\]

\[
\hat{r} = \frac{r}{|z|^2 - r^2}
\]

\[
|z|^2 - r^2 = \frac{r}{\hat{r}}
\]

\[
\frac{|z|^2}{r^2} - 1 = \frac{1}{\hat{r}r} = \hat{bb}
\]

\[
\hat{bb} - |bz|^2 = -1
\]
Crystallographic Sphere Packings

- First introduced by Kontorovich & Nakamura in 2017
Crystallographic Sphere Packings

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- A **crystallographic sphere packing** is generated by the action of a *geometrically finite* reflection group
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Crystallographic Sphere Packings

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  - **Geometrically finite**: generated by a finite number of fundamental reflections
  - Groups that are geometrically finite have a finite **fundamental polytope**, or the region bounded by the planes associated with their fundamental reflections
  - The fundamental polytope encodes the same information as a **Coxeter diagram**

![Coxeter diagram](image)
A Coxeter diagram is a collection of nodes and edges that represents a geometric relationship between n-dimensional spheres and hyperplanes. For two nodes $i, j$, the edge $e_{i,j}$ is defined by the following:

$$e_{i,j} = \begin{cases} 
\text{a dotted line,} & \text{if } i \text{ and } j \text{ are disjoint} \\
\text{a thick line,} & \text{if } i \text{ and } j \text{ are tangent} \\
\text{m − 2 thin lines,} & \text{if the angle between } i \text{ and } j \text{ is } \pi/m \\
\text{no line,} & \text{if } i \perp j 
\end{cases}$$
Computation of the Coxeter Diagram
Computation of the Coxeter Diagram
Computation of the Coxeter Diagram

[Diagram showing a Coxeter diagram with nodes labeled 1, 2, 3, 4, 5 and connections between nodes.]
Computation of the Coxeter Diagram
Computation of the Coxeter Diagram
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Computation of the Coxeter Diagram
Computation of the Coxeter Diagram
Computation of the Coxeter Diagram
Computation of the Coxeter Diagram
Computation of the Coxeter Diagram
Cluster and Cocluster

In a Coxeter diagram, we select nodes that are connected to each other only by thick or dashed lines, and to the rest by thick or dashed lines, or no lines at all. For instance:

In each case, the selected nodes form the isolated cluster, and the remainder is the cocluster.
Cluster and Cocluster

The cocluster acts on the cluster through sphere inversions.
Cluster and Cocluster

The cocluster acts on the cluster through sphere inversions. Eerily enough, we get packings!
Cluster and Cocluster

The cocluster acts on the cluster through sphere inversions. Eerily enough, we get packings!
Structure Theorem

This is no coincidence.
Structure Theorem

This is no coincidence. In 2017, Kontorovich and Nakamura proved the **Structure Theorem for crystallographic packings**: a Coxeter diagram’s isolated cluster generates a crystallographic packing in this manner, and all crystallographic packings arise as the orbit of an isolated cluster.
Finiteness Theorem

Why are crystallographic sphere packings a pressing topic? Recently, Kontorovich and Nakamura proved that there exist finitely many crystallographic packings. In fact, no such packings exist in higher than 21 dimensions.
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This means that crystallographic packings can be systematically explored and classified — which was a large part of our research this summer.
Finiteness Theorem

Why are crystallographic sphere packings a pressing topic? Recently, Kontorovich and Nakamura proved that there exist finitely many crystallographic packings. In fact, no such packings exist in higher than 21 dimensions. This means that crystallographic packings can be systematically explored and classified – which was a large part of our research this summer.

There are 3 sources that can be used to generate crystallographic packings, and each of us focused on one source:

- Alisa – Polyhedra
- Devora – Bianchi groups
- Zack – Higher dimensional quadratic forms
Sources of Circle Packings

Polyhedra
- Apply K-A-T Theorem

Dimension $n \geq 3$
- Select quadratic form
  - Apply Vinberg's algorithm
  - Obtain fundamental polyhedron
    - Describe with Coxeter diagram
  
Bianchi groups

Apply Structure Theorem
- Generate circle packing

Which are integral?
Polyhedra

- How can circle packings arise from polyhedra?
Theorem: Every polyhedron (up to combinatorial equivalence) has a midsphere.
Polyhedra: Koebe-Andreev-Thurston Theorem

Theorem: Every polyhedron (up to combinatorial equivalence) has a midsphere.

Combinatorial equivalence: containing the same information about faces, edges, and vertices, regardless of physical representation.
Polyhedra: Koebe-Andreev-Thurston Theorem

- Theorem: Every polyhedron (up to combinatorial equivalence) has a midsphere.
  - Combinatorial equivalence: containing the same information about faces, edges, and vertices, regardless of physical representation
  
- Midsphere: a sphere tangent to every edge in a polyhedron
Polyhedra: Koebe-Andreev-Thurston Theorem

- The midsphere gives rise to two sets of circles: **facet circles** (purple) and **vertex horizon circles** (pink)

Planar representation of a polyhedron (left), its vertex horizon circles (center), and its realization with midsphere, vertex horizon circles, and facet circles (right).

Source: David Eppstein 2004
Polyhedra: Koebe-Andreev-Thurston Theorem

- **Stereographically projecting** the facet and vertex horizon circles onto $\mathbb{R}^2$ yields a collection of circles in the plane.
  - Stereographic projections map a sphere onto the plane, preserving tangencies and angles

Source: Strebe
Polyhedra: Koebe-Andreev-Thurston Theorem

- **Stereographically projecting** the facet and vertex horizon circles onto $\mathbb{R}^2$ yields a collection of circles in the plane.
  - Stereographic projections map a sphere onto the plane, preserving tangencies and angles.

Source: David E. Joyce 2002
Polyhedra: Koebe-Andreev-Thurston Theorem
By K-A-T, this collection of circles is unique up to circle inversions
By K-A-T, this collection of circles is unique up to circle inversions.

These circles actually generate a packing: let pink $\rightarrow$ cluster, purple $\rightarrow$ cocluster.
Polyhedra: Generating a Packing

By constructing the Coxeter diagram of this cluster/co-cluster group, we can see that the Structure Theorem applies
Polyhedra: Generating a Packing

By constructing the Coxeter diagram of this cluster/cocluster group, we can see that the Structure Theorem applies.
By constructing the Coxeter diagram of this cluster/cocluster group, we can see that the Structure Theorem applies.
Polyhedra: Generating a Packing
Polyhedra: Methods

- Polyhedron data was generated with **plantri**, a program created by Brinkmann and McKay
- We wrote code in Mathematica using some techniques from Ziegler 2004
- Data is being collected and presented on our website
## Polyhedral Circle Packings

<table>
<thead>
<tr>
<th>Polyhedron</th>
<th>Strip Supercluster</th>
<th>Strip Packing</th>
<th>Gramian Matrix</th>
<th>Inversive Coordinates</th>
<th>Bend Matrices</th>
<th>Mathematica File</th>
</tr>
</thead>
</table>
| Tetrahedron      | [Image]            | [Image]       | \[
\begin{pmatrix}
-1 & 1 & 1 & 1 & 0 & 0 & 0 & 2 \\
1 & -1 & 1 & 1 & 0 & 0 & 2 & 0 \\
1 & 1 & -1 & 1 & 0 & 2 & 0 & 0 \\
1 & 1 & 1 & -1 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & -1 & 1 & 1 & 1 \\
0 & 0 & 2 & 0 & 0 & 0 & -1 & 1 \\
0 & 2 & 0 & 0 & 0 & 1 & 1 & -1 \\
2 & 0 & 0 & 0 & 0 & 1 & 1 & -1
\end{pmatrix}
\] | [Code] | Integral       |                |
| Square Pyramid   | [Image]            | [Image]       | \[
\begin{pmatrix}
-1 & 1 & 3 & 1 & 1 & 0 & 2\sqrt{2} & 0 & 2\sqrt{2} & 0 \\
1 & -1 & 1 & 3 & 1 & 0 & 0 & 0 & 2\sqrt{2} & 2\sqrt{2} \\
3 & 1 & -1 & 1 & 1 & 2\sqrt{2} & 0 & 0 & 0 & 2\sqrt{2} \\
1 & 3 & 1 & -1 & 1 & 2\sqrt{2} & 2\sqrt{2} & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & -1 & 0 & 0 & \sqrt{2} & 0 & 0 \\
0 & 0 & 2\sqrt{2} & 2\sqrt{2} & 0 & -1 & 1 & 1 & 0 & 3 \\
2\sqrt{2} & 0 & 0 & 2\sqrt{2} & 0 & 1 & -1 & 1 & 1 & 3 \\
0 & 0 & 0 & 0 & 0 & \sqrt{2} & 1 & 1 & -1 & 1 \\
2\sqrt{2} & 2\sqrt{2} & 0 & 0 & 0 & 3 & 1 & 1 & -1 & 1 \\
0 & 2\sqrt{2} & 2\sqrt{2} & 0 & 0 & 1 & 3 & 1 & 1 & -1
\end{pmatrix}
\] | [Code] | Integral       |                |
Polyhedra: Findings

Interested in which polyhedra give rise to integral packings
Polyhedra: Findings

- Previously known integral polyhedra: tetrahedron, square pyramid, hexagonal pyramid, and gluings thereof
Polyhedra: Findings

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  - Define a **gluing** operation to be a joining along vertices or faces
Polyhedra: Findings

▶ Previously known integral polyhedra: tetrahedron, square pyramid, hexagonal pyramid, and gluings thereof
  ▶ Define a **gluing** operation to be a joining along vertices or faces
▶ Since the tetrahedron, square pyramid, and hexagonal pyramid cannot be decomposed (unglued) into smaller integral polyhedra, they can be considered **seed polyhedra**
Polyhedra: Findings

- Previously known integral polyhedra: tetrahedron, square pyramid, hexagonal pyramid, and gluings thereof
  - Define a **gluing** operation to be a joining along vertices or faces
- Since the tetrahedron, square pyramid, and hexagonal pyramid cannot be decomposed (unglued) into smaller integral polyhedra, they can be considered **seed polyhedra**
- We found a new integral seed polyhedron!
Polyhedra: Findings - 6v7f_2
This is the packing of a hexagonal pyramid; it is in fact the same packing as 6v7f_2.
Polyhedra: Findings - 6v7f_2
Bianchi groups, Bi($m$), are the set of 2x2 matrices whose entries are of the complex form $a + b\sqrt{-m}$, and which have determinant 1.
Bianchi Groups

Bianchi groups, Bi($m$), are the set of 2x2 matrices whose entries are of the complex form $a + b\sqrt{-m}$, and which have determinant 1. Luigi Bianchi began studying these groups over 100 years ago, in 1892...
Bianchi Groups

Bianchi groups, Bi$(m)$, are the set of 2x2 matrices whose entries are of the complex form $a + b\sqrt{-m}$, and which have determinant 1. Luigi Bianchi began studying these groups over 100 years ago, in 1892... Before they were ever connected to circle packings!
Sui gruppi di sostituzioni lineari con coefficienti appartenenti a corpi quadratici immaginari.

Di

LUIGI BIANCHI a Pisa.

Prefazione.

La presente Memoria tratta dei gruppi di sostituzioni lineari:

\( z' = \frac{\alpha z + \beta}{\gamma z + \delta} \)

sopra una variabile complessa \( z \), i cui coefficienti \( \alpha, \beta, \gamma, \delta \) percorrono tutti i numeri interi di un corpo quadratico immaginario \( \Omega \), assoggettati alla sola condizione

\[(2) \alpha \delta - \beta \gamma = 1. *)\]

Essa è una continuazione del lavoro da me pubblicato nel Vol. XXXVIII di questi Annali, ove già è indicata la generalizzazione, che qui trova il suo effettivo svolgimento. **)

Ogni numero intero o frazionario in \( \Omega \) ha la forma:

\[(3) \quad m + in\sqrt{D}, \]
Bianchi Groups

Figure: Bi(2): From 1892 to 2018
Bianchi was interested in exploring which Bianchi groups are *reflective*, meaning finitely generated. 120 years later, Belolipetsky and McLeod conclusively showed that there are a finite number of these reflective Bianchi groups, and enumerated them.
Bianchi Groups

Bianchi was interested in exploring which Bianchi groups are reflective, meaning finitely generated. 120 years later, Belolipetsky and McLeod conclusively showed that there are a finite number of these reflective Bianchi groups, and enumerated them.

The reflective Bianchi groups can be used to generate circle packings. But how do we go from matrices to circles?
Bianchi Groups

Apply K-A-T Theorem

Select quadratic form

Apply Vinberg's algorithm

Obtain fundamental polyhedron

Describe with Coxeter diagram

Apply Structure Theorem

Generate circle packing

Which are integral?
Bianchi Groups

- Polyhedra
  - Apply K-A-T Theorem
- Dimension $n \geq 3$
  - Select quadratic form
  - Apply Vinberg’s algorithm
  - Obtain fundamental polyhedron
  - Describe with Coxeter diagram
- Bianchi groups
  - Apply Structure Theorem
  - Generate circle packing
  - Which are integral?

McLeod, Vinberg
Bianchi Groups

This summer, using McLeod’s application of Vinberg’s algorithm, we catalogued all known circle packings that arise from Bianchi groups.
This summer, using McLeod’s application of Vinberg’s algorithm, we catalogued all known circle packings that arise from Bianchi groups.

**Polyhedral Circle Packings**

Click to expand

**Bianchi Group Packings**

Click to expand

\[-a_0^2 + \sum_{i=1}^{n} a_i^2\]

Click to expand

\[-2a_0^2 + \sum_{i=1}^{n} a_i^2\]

Click to expand

\[-3a_0^2 + \sum_{i=1}^{n} a_i^2\]

Click to expand
Bianchi Groups

This summer, using McLeod’s application of Vinberg’s algorithm, we catalogued all known circle packings that arise from Bianchi groups.

Polyhedral Circle Packings

Click to expand

Bianchi Group Packings

Click to expand

<table>
<thead>
<tr>
<th>Group</th>
<th>Visualization</th>
<th>Coxeter Diagram</th>
<th>Strip Packings</th>
<th>Gramian Matrix</th>
<th>Inversive Coordinates</th>
<th>Bend Matrices</th>
<th>Mathematica File</th>
</tr>
</thead>
</table>
| B(1)   | ![Visual](image1) | ![Diagram](image2) | ![Packings](image3) | \[
\begin{pmatrix}
-1 & 1 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & \frac{1}{2} \\
0 & 0 & -1 & 1 & 0 \\
0 & 0 & 1 & -1 & \frac{1}{2} \\
0 & \frac{1}{2} & 0 & \frac{1}{2} & -1
\end{pmatrix}
\] | | | Code |
| B(2)   | ![Visual](image4) | ![Diagram](image5) | ![Packings](image6) | \[
\begin{pmatrix}
-1 & 1 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & \frac{1}{2} \\
0 & 0 & -1 & 1 & 0 \\
0 & 0 & 1 & -1 & \frac{1}{2} \\
0 & \frac{1}{2} & 0 & \frac{1}{2} & -1
\end{pmatrix}
\] | | | Code |
| B(3)   | ![Visual](image7) | ![Diagram](image8) | ![Packings](image9) | \[
\begin{pmatrix}
-1 & \frac{1}{2} & \frac{1}{2} & 0 \\
\frac{1}{2} & -1 & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\
0 & \frac{1}{2} & \frac{1}{2} & 0
\end{pmatrix}
\] | | None | Code |
Bianchi Groups and Integrality

An interesting property of Bianchi group circle packings is that most of them are \textit{integral}. 

\begin{center}
\begin{tabular}{c|c|c|c|c}
 & & & & \\
1 & 2 & 2 & 2 & 2 \\
2 & 3 & 3 & 3 & 3 \\
2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 & 1 \\
3 & 3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 & 4 \\
3 & 3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 & 4 \\
\end{tabular}
\end{center}
Bianchi Groups and Integrality

An interesting property of Bianchi group circle packings is that most of them are *integral*.

It’s very easy to see that a packing is integral empirically just by computing the bends of the packing, but there’s actually a way to *prove* integrality more rigorously.
An interesting property of Bianchi group circle packings is that most of them are integral.

It’s very easy to see that a packing is integral empirically just by computing the bends of the packing, but there’s actually a way to prove integrality more rigorously. Likewise, there’s a way to prove nonintegrality of a packing.
Bianchi Groups and Integrality

An interesting property of Bianchi group circle packings is that most of them are integral.

It’s very easy to see that a packing is integral empirically just by computing the bends of the packing, but there’s actually a way to prove integrality more rigorously.

Likewise, there’s a way to prove nonintegrality of a packing.

An exciting part of our work this summer is proving integrality and nonintegrality for all known Bianchi packings.
One note, which will also be relevant shortly, is that an insight in Kontorovich & Nakamura’s 2017 paper was the observation that what was thought to be the $\hat{Bi}(3)$ Coxeter diagram did not actually represent the full group of mirrors:
Doubling in $\hat{Bi}(3)$
Doubling in $\hat{Bi}(3)$
Doubling in $\hat{Bi}(3)$

Through a further series of operations, we can transform the diagram into the diagram.
Doubling in $\hat{Bi}(3)$

Through a further series of operations, we can transform the diagram $\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array}$ into the diagram $\begin{array}{cccc} 1 & 2 & 3 & 4 & 5 \end{array}$. However, this was done less systematically; it primarily derived from looking at the orbit of the original generators acting on themselves until a valid configuration was found that has an isolated cluster.
Questions About Existence of Packings

Now that we’ve covered polyhedra and Bianchi groups, which give all the crystallographic packings in two dimensions, the natural question is: Do higher dimensional packings exist? What can we say about them?
Questions About Existence of Packings

Now that we’ve covered polyhedra and Bianchi groups, which give all the crystallographic packings in two dimensions, the natural question is: Do higher dimensional packings exist? What can we say about them? We can answer this by looking at Coxeter diagrams of higher-dimensional configurations, and applying the Structure Theorem, which still holds.
High-Dimensional Packings

This summer, I worked on putting together the known candidates for these packings, namely 41 potential packings for quadratic forms $-dx_0^2 + \sum_{i=1}^{n} x_i^2$ for $d = 1, 2, 3$. 
This summer, I worked on putting together the known candidates for these packings, namely 41 potential packings for quadratic forms $-dx_0^2 + \sum_{i=1}^{n} x_i^2$ for $d = 1, 2, 3$. Here is a snapshot of how some appear on our website:

$$-2x_0^2 + \sum_{i=1}^{n} x_i^2$$
High-Dimensional Packings

The techniques discussed here have also allowed us to attack the following diagrams that lack isolated clusters:

Source: John McLeod
High-Dimensional Packings

The techniques discussed here have also allowed us to attack the following diagrams that lack isolated clusters:

![Diagrams]

Source: John McLeod

What’s something that all of these have in common?
High-Dimensional Packings

The techniques discussed here have also allowed us to attack the following diagrams that lack isolated clusters:

What’s something that all of these have in common? They all feature 1-2-3-4 as a subdiagram!
So, if we apply the known transformation for $\begin{array}{c}
  1 \\
  \hline
  2 \\
  3 \\
  4 \\
\end{array}$ followed by a suitable action on the remainder of the nodes in the Coxeter diagram, then hopefully we will obtain a valid diagram representing one such desired subgroup of mirrors.
Results

The following Coxeter diagrams were obtained for the $n = 6, 7, 8$ cases of the quadratic form $-3x_0^2 + \sum_{i=1}^{n} x_i^2$:
The following is believed to work for $n = 10$, and works for $n = 11$:
Lastly, this behemoth is a diagram for $n = 13$: 

\[\begin{array}{cccccccccccccccc}
\end{array}\]
We are much indebted to the following papers:

- L. Bianchi, “Sui gruppi di sostituzioni lineari con coefficienti appartenenti a corpi quadratici immaginari,” 1892.
Acknowledgements

We would like to acknowledge support and funding from: NSF DMS-1802119; DIMACS; and the Rutgers Mathematics Department. We would also like to thank Kei Nakamura and Alice Mark for taking the time to discuss this material with us; and we would most of all like to thank Professor Kontorovich for his guidance and mentorship on these projects.