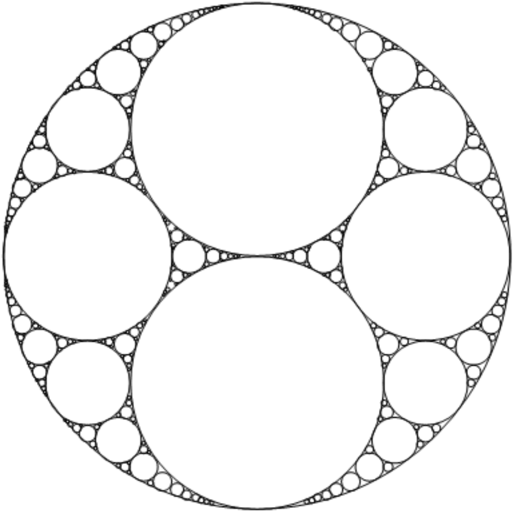
DIMACS REU 2018
Project: Sphere Packings and Number Theory

Alisa Cui, Devora Chait, Zachary Stier
Mentor: Prof. Alex Kontorovich

July 13, 2018

Apollonian Circle Packing

This is an Apollonian circle packing:



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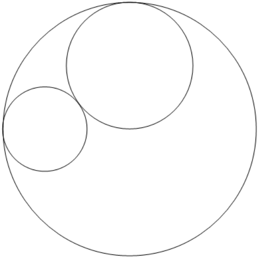

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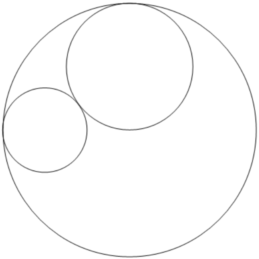

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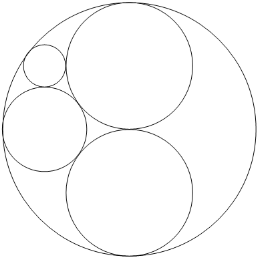
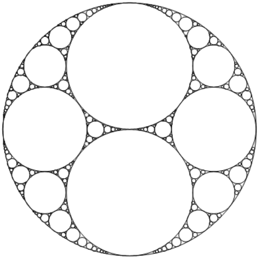


Apollonian Circle Packing

- Start with three mutually tangent circles
- Draw two more circles, each of which is tangent to the original three
- Continue drawing tangent circles, densely filling space



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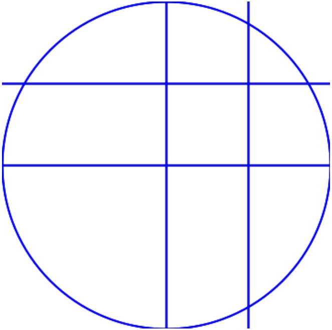

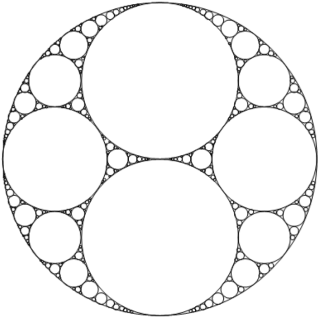

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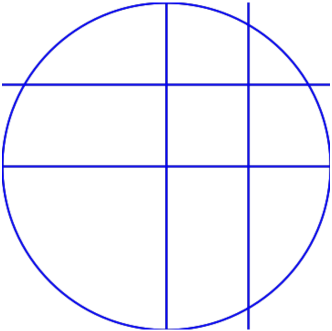

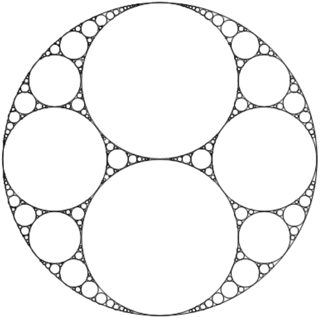

These two images actually represent the same circle packing!

We can go from one realization to the other using **circle inversions**.

Circle Inversions

Circle inversion sends points at a distance of rd from the center of the mirror circle to a distance of r/d from the center of the mirror circle.

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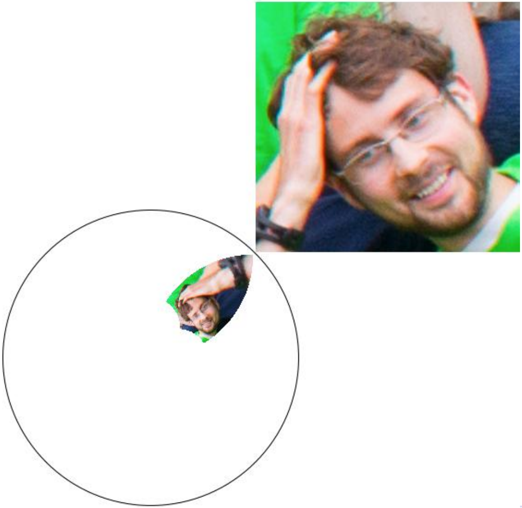
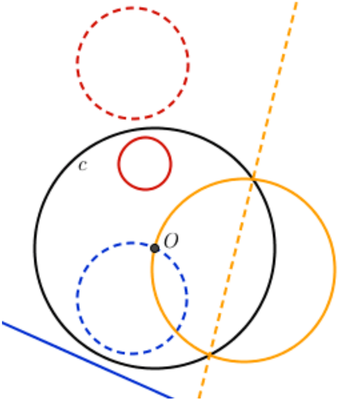


Circle Inversions

Circle inversion sends points at a distance of rd from the center of the mirror circle to a distance of r/d from the center of the mirror circle.

- We apply circle inversions to both spheres and planes, where planes are considered spheres of infinite radius.
- Circle inversions preserve tangencies and angles.



Source: Malin Christersson

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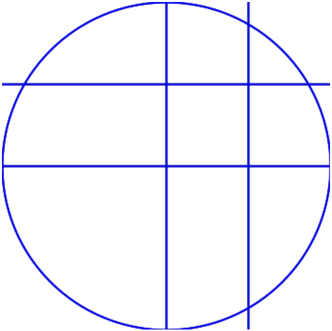

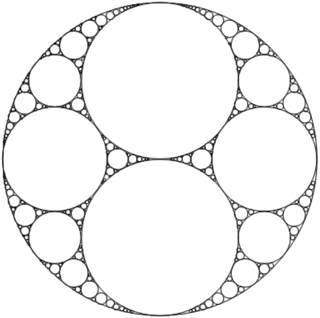

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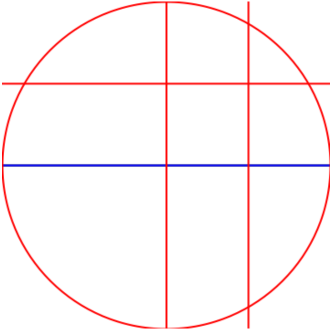

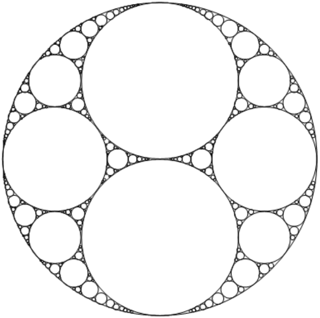

To generate a packing, invert the blue line about the reds

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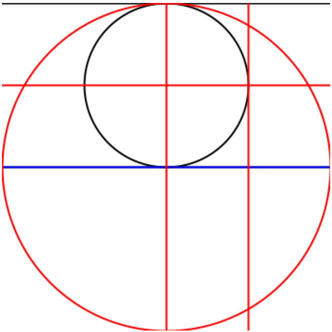

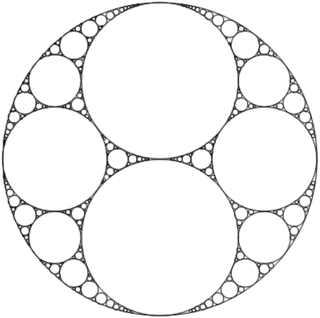

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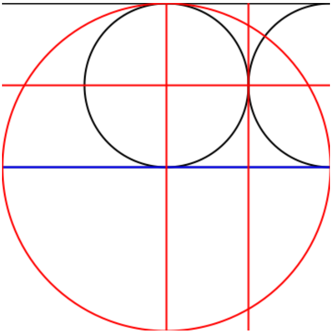

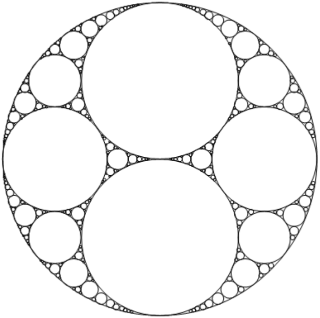

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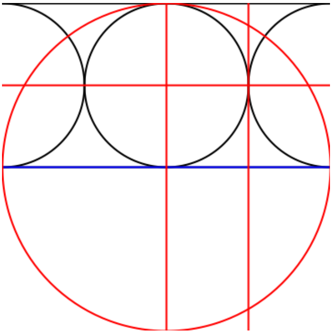

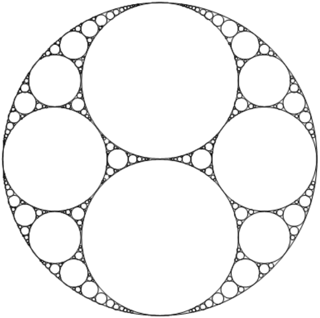

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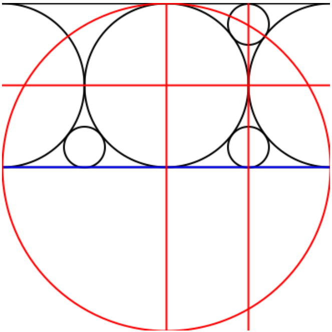

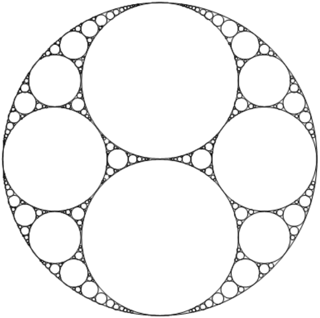

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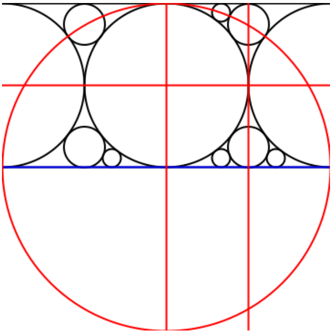

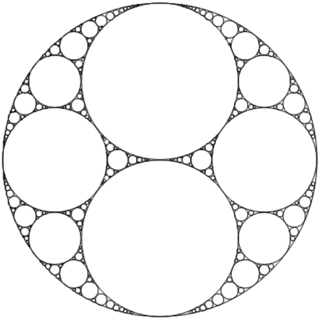

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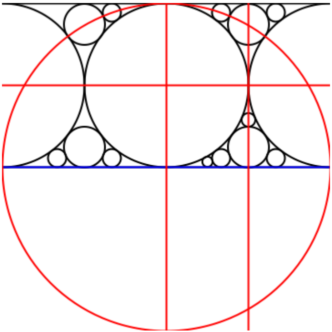 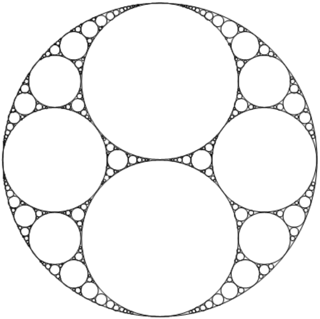

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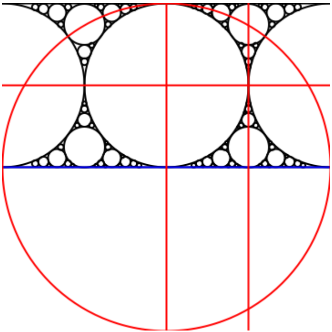

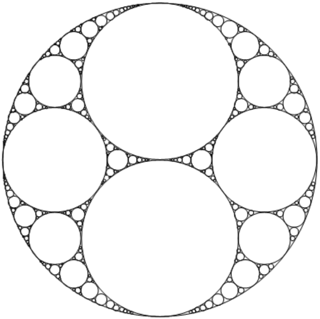

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Sphere Packings: Definition

The sphere packings we've examined this summer are configurations where the spheres:

- have varying radii
- are oriented to have mutually disjoint interiors
- densely fill up space

Hyperbolic Geometries

- There is a surprising connection between sphere packings and non-Euclidean geometries.

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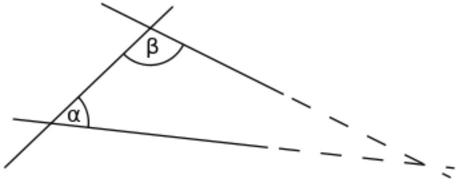
- Euclidean geometry is characterized by Euclid's *parallel postulate*, which states that the angles formed by two lines intersecting on one side of a third line sum to be less than π radians.



Source: Wikipedia

Hyperbolic Geometries

- These geometries have several models which are each used as is necessary.
- For now, we are going to focus on the **upper half-space model** of $\mathbb{H}^{n+1}$: consider $\mathbb{R}^{n+1}$, subject to $x_0 > 0$. This space has its own metric, and has as its boundary $\mathbb{R}^n$.

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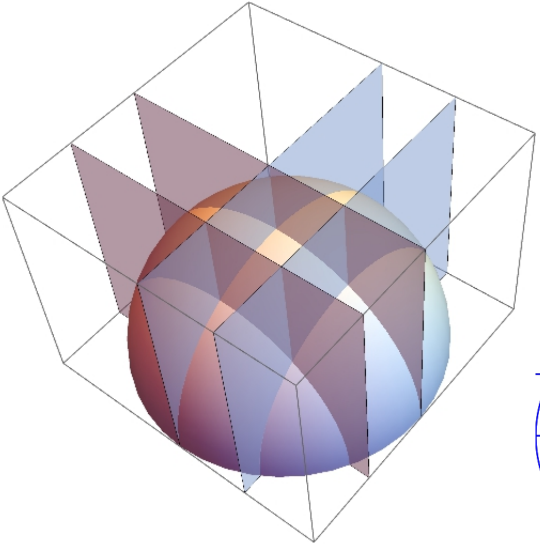
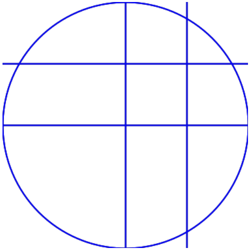
- Because of the different metric, planes in $\mathbb{H}^{n+1}$ are actually hemispheres, with their circumferences lying in $\mathbb{R}^n$ (i.e., the subset $x_0 = 0$).
- Conveniently, we've already been looking at spheres lying in $\mathbb{R}^n$! We can "continue our configurations upwards" in what is known as the **Poincaré extension**.

Poincaré Extension





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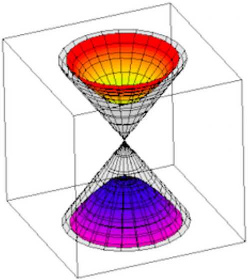

Resting in $\mathbb{R}^3$
Source: supermath.info

- A quadratic form Q is a polynomial where each term is of degree exactly 2. It can be used to define an inner product space.
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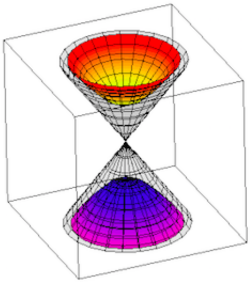

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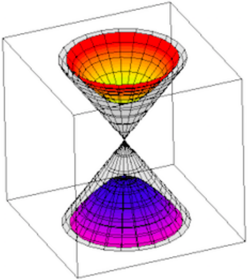

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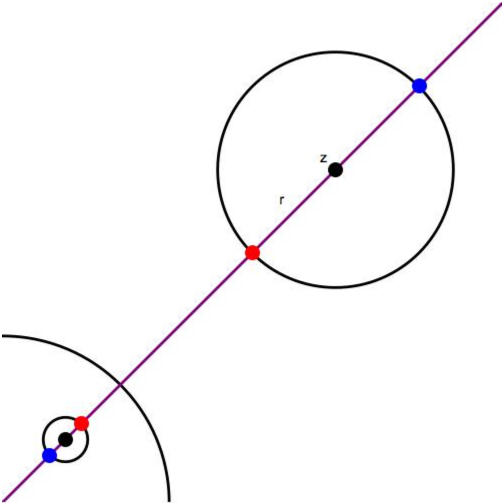
Where did this quadratic form $Q = -1$ come from? Circle inversions!

From Circle Inversions to Quadratic Forms



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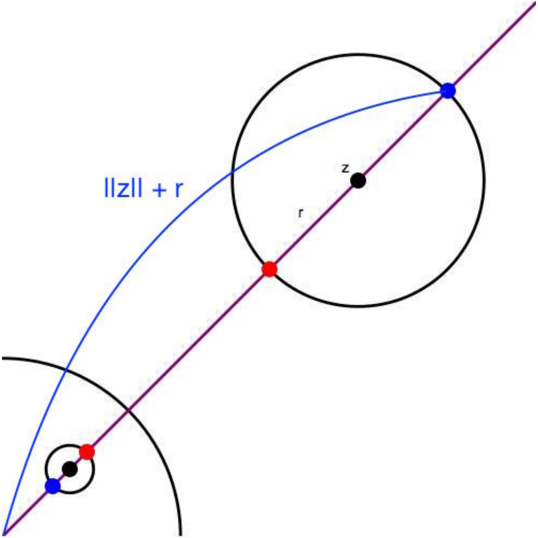

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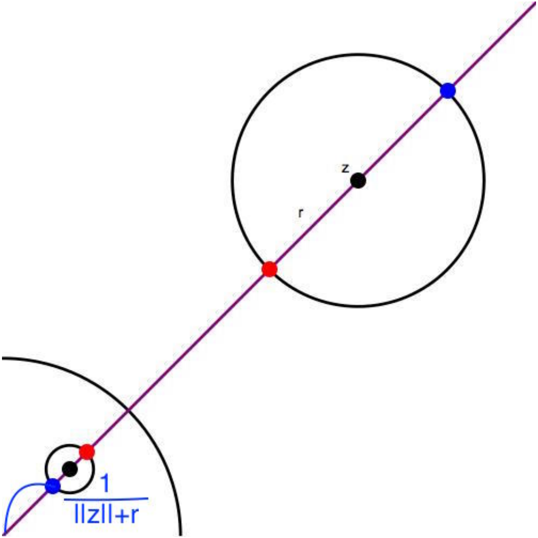

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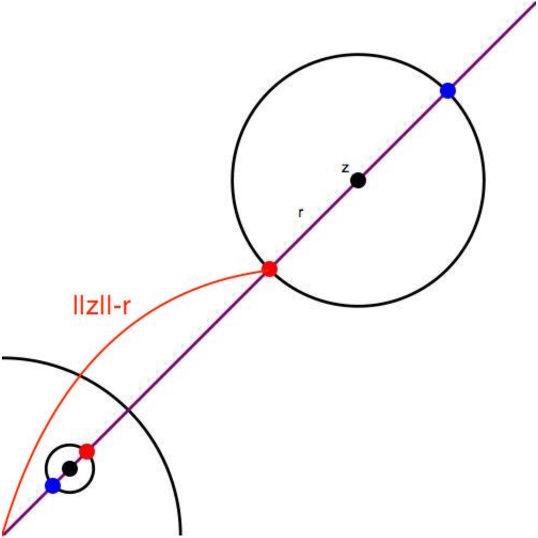

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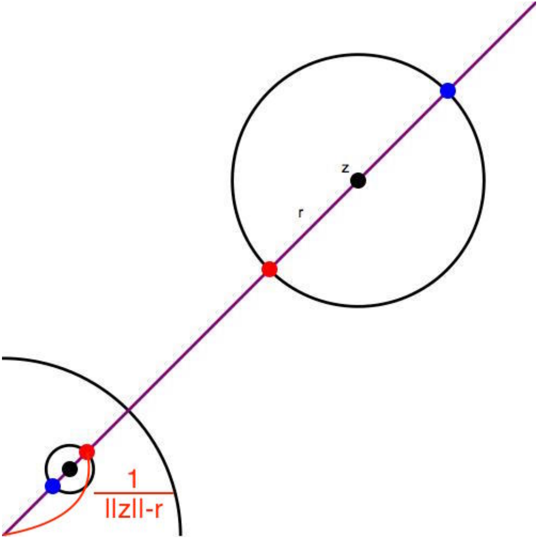

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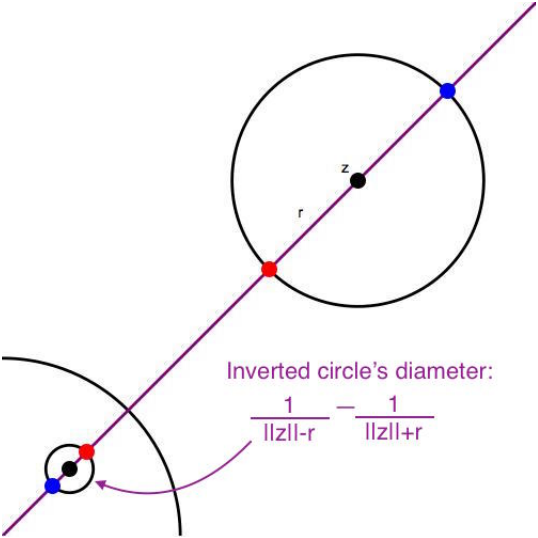

From Circle Inversions to Quadratic Forms

$$\hat{d} = \frac{1}{|z| - r} - \frac{1}{|z| + r}$$

$$\hat{d} = \frac{2r}{|z|^2 - r^2}$$

$$\hat{r} = \frac{r}{|z|^2 - r^2}$$

$$|z|^2 - r^2 = \frac{r}{\hat{r}}$$

$$\frac{|z|^2}{r^2} - 1 = \frac{1}{\hat{r}r} = \hat{b}b$$

$$\hat{\mathbf{b}}\mathbf{b} - |\mathbf{bz}|^2 = -\mathbf{1}$$

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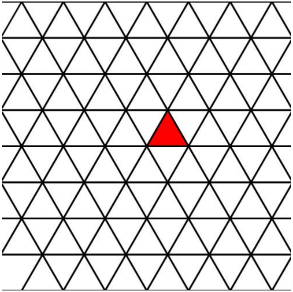
  - Groups that are geometrically finite have a finite **fundamental polytope**, or the region bounded by the planes associated with their fundamental reflections
  - The fundamental polytope encodes the same information as a **Coxeter diagram**

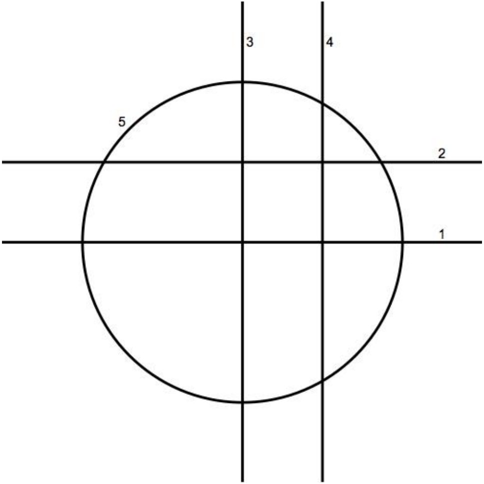


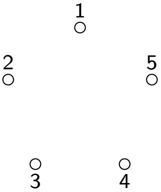
Coxeter Diagram

A **Coxeter diagram** is a collection of nodes and edges that represents a geometric relationship between n-dimensional spheres and hyperplanes. For two nodes i, j, the edge $e_{i,j}$ is defined by the following:

$$e_{i,j} = \begin{cases} \text{a dotted line,} & \text{if } i \text{ and } j \text{ are disjoint} \\ \text{a thick line,} & \text{if } i \text{ and } j \text{ are tangent} \\ m-2 \text{ thin lines,} & \text{if the angle between } i \text{ and } j \text{ is } \pi/m \\ \text{no line,} & \text{if } i \perp j \end{cases}$$

Computation of the Coxeter Diagram





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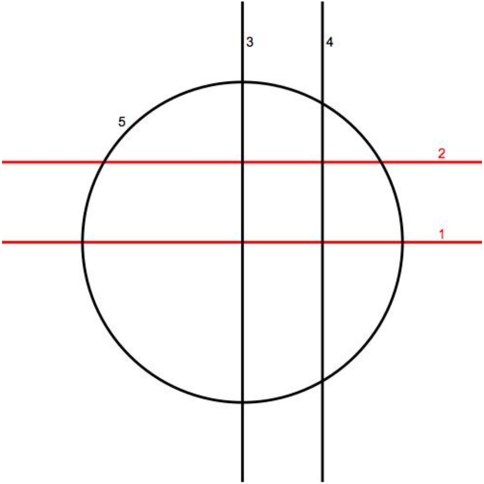

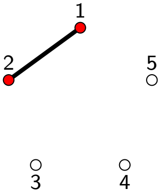

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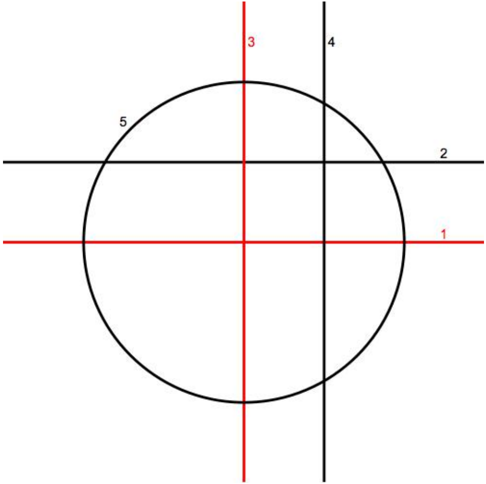

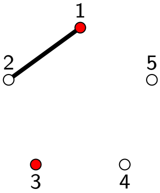

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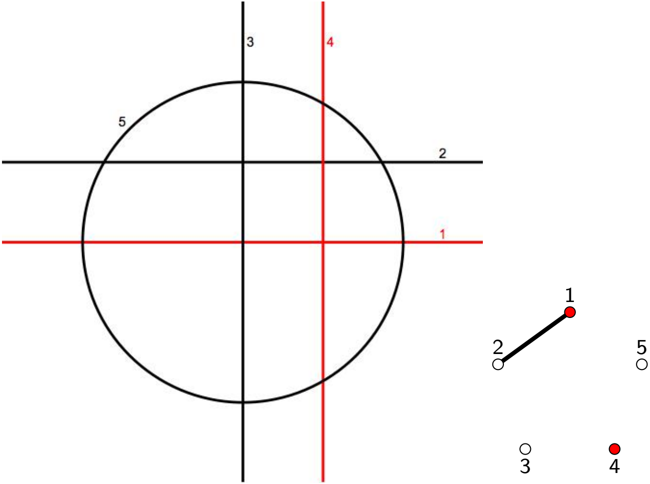

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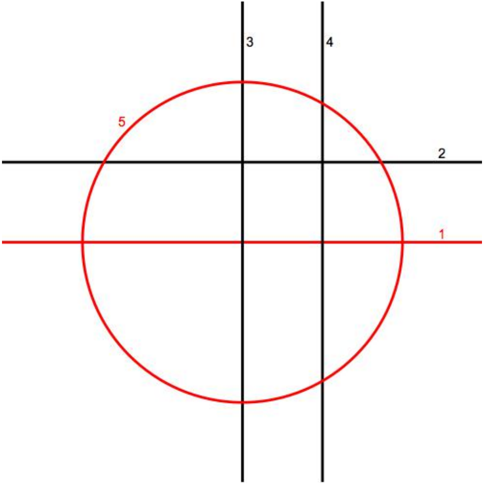

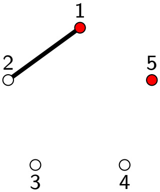

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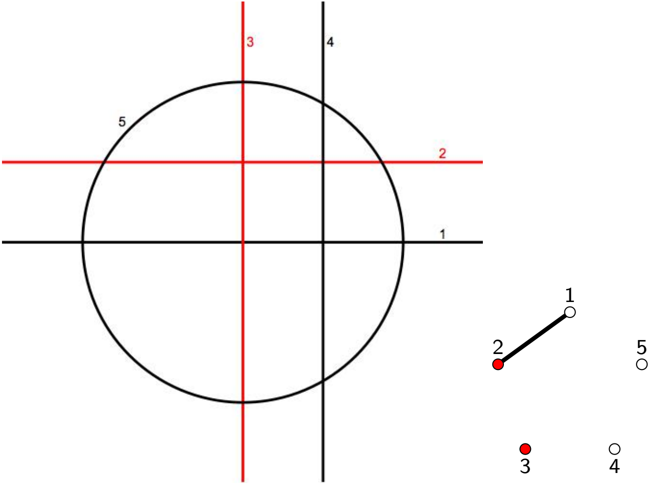

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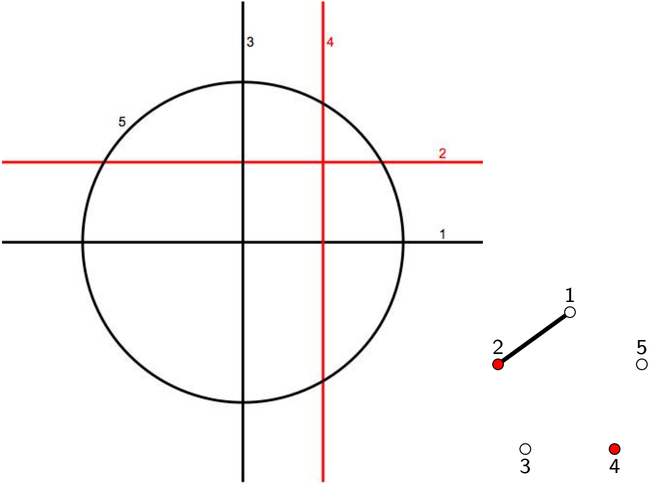

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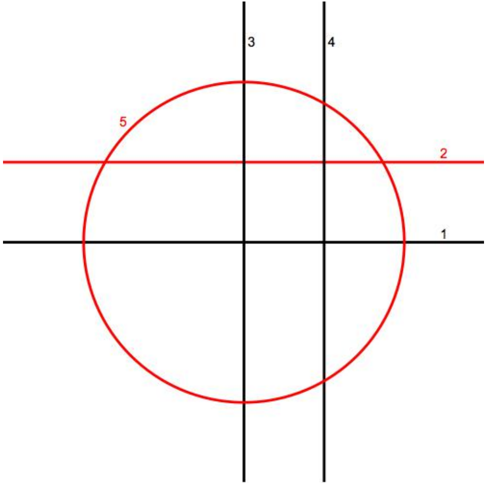

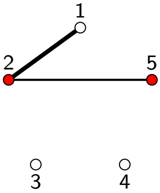

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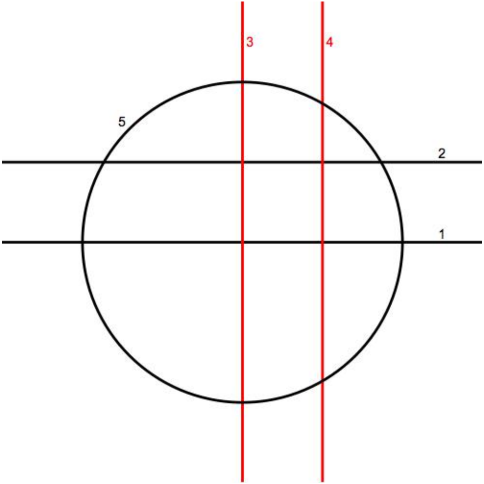

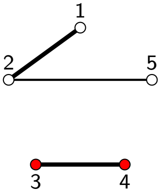

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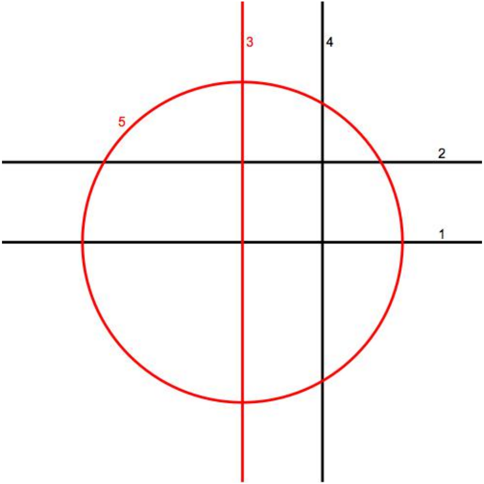

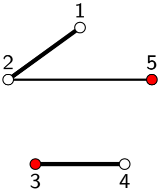

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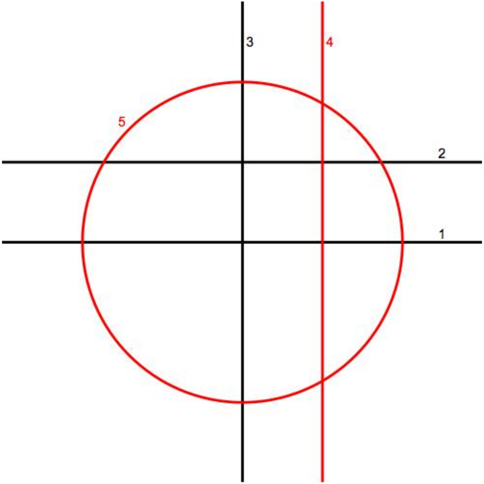

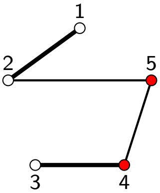

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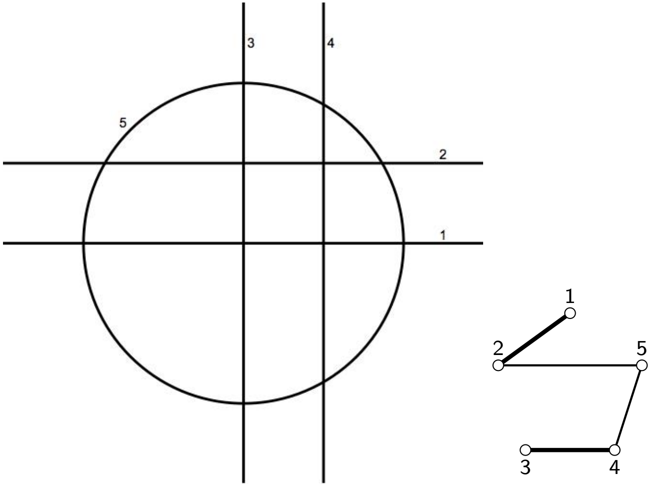
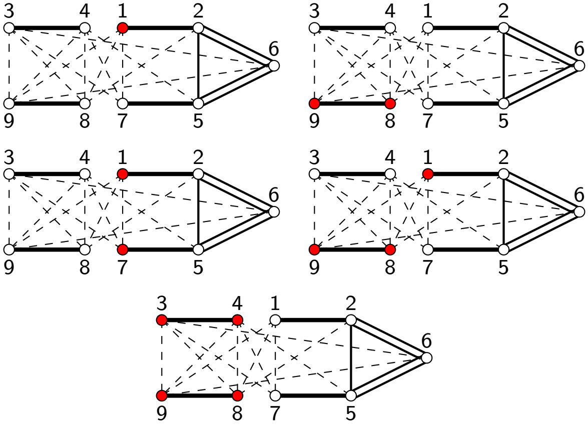


Cluster and Cocluster

In a Coxeter diagram, we select nodes that are connected to each other only by thick or dashed lines, and to the rest by thick or dashed lines, or no lines at all. For instance:



In each case, the selected nodes form the **isolated cluster**, and the remainder is the **cocluster**.

Cluster and Cocluster

The cocluster acts on the cluster through sphere inversions.

Cluster and Cocluster

The cocluster acts on the cluster through sphere inversions. Eerily enough, we get packings!

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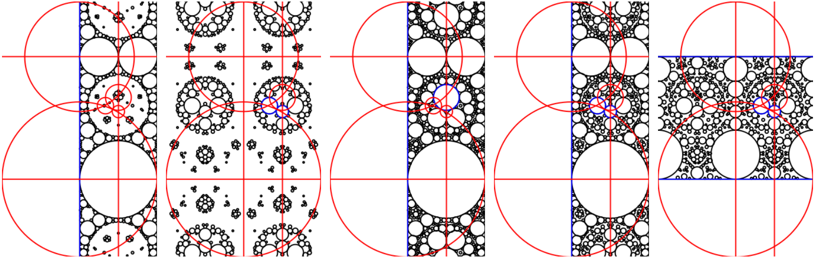

Structure Theorem

This is no coincidence.

Structure Theorem

This is no coincidence. In 2017, Kontorovich and Nakamura proved the **Structure Theorem for crystallographic packings**: a Coxeter diagram's isolated cluster generates a crystallographic packing in this manner, and all crystallographic packings arise as the orbit of an isolated cluster.

Finiteness Theorem

Why are crystallographic sphere packings a pressing topic? Recently, Kontorovich and Nakamura proved that there exist finitely many crystallographic packings. In fact, no such packings exist in higher than 21 dimensions.

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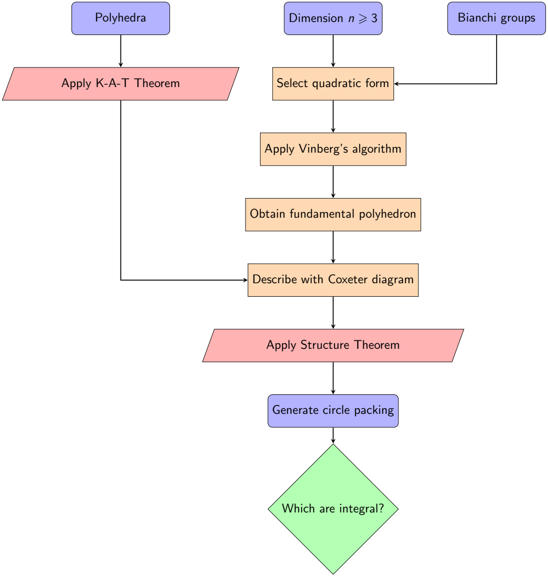
This means that crystallographic packings can be systematically explored and classified – which was a large part of our research this summer.

There are 3 sources that can be used to generate crystallographic packings, and each of us focused on one source:

- Alisa – Polyhedra
- Devora – Bianchi groups
- Zack – Higher dimensional quadratic forms

Sources of Circle Packings



Polyhedra

- How can circle packings arise from polyhedra?

Polyhedra: Koebe-Andreev-Thurston Theorem

- Theorem: Every polyhedron (up to **combinatorial equivalence**) has a **midsphere**.

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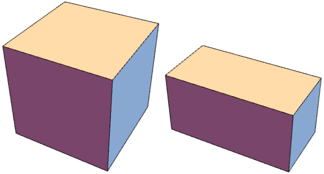

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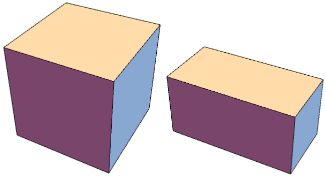
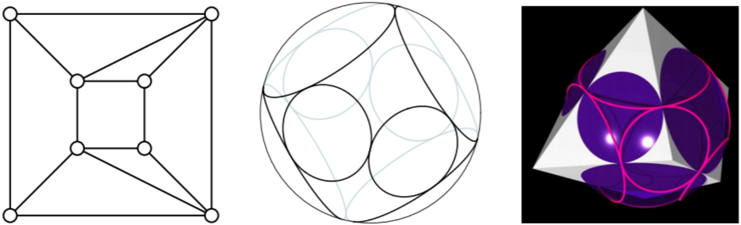


  - Midsphere: a sphere tangent to every edge in a polyhedron

Polyhedra: Koebe-Andreev-Thurston Theorem

- The midsphere gives rise to two sets of circles: **facet circles** (purple) and **vertex horizon circles** (pink)

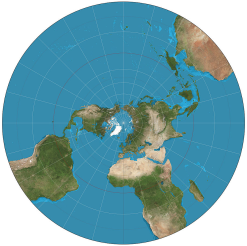


Planar representation of a polyhedron (left), its vertex horizon circles (center), and its realization with midsphere, vertex horizon circles, and facet circles (right).

Source: David Eppstein 2004

Polyhedra: Koebe-Andreev-Thurston Theorem

- **Stereographically projecting** the facet and vertex horizon circles onto $\mathbb{R}^2$ yields a collection of circles in the plane.
  - Stereographic projections map a sphere onto the plane, preserving tangencies and angles



Source: Strebe

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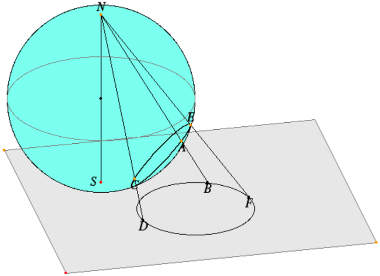
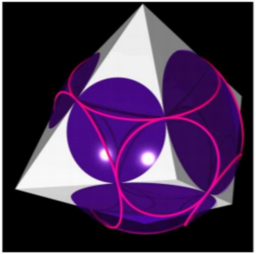
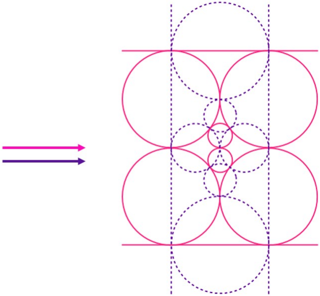


Source: David E. Joyce 2002

Polyhedra: Koebe-Andreev-Thurston Theorem





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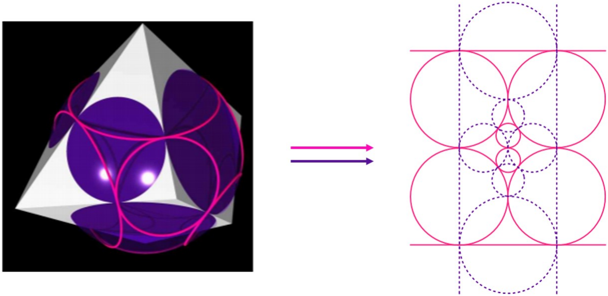

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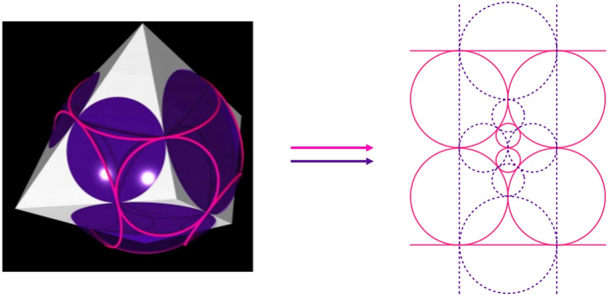
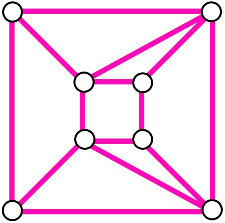
Polyhedra: Koebe-Andreev-Thurston Theorem

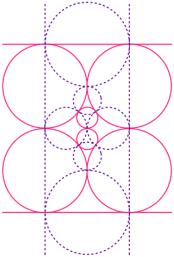


- By K-A-T, this collection of circles is unique up to circle inversions
- These circles actually generate a packing: let pink $\rightarrow$ cluster, purple $\rightarrow$ cocluster

Polyhedra: Generating a Packing

By constructing the Coxeter diagram of this cluster/cocluster group, we can see that the Structure Theorem applies





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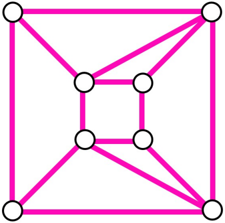

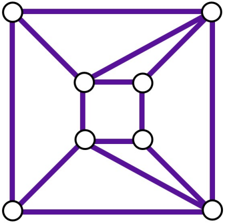

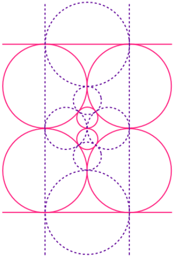

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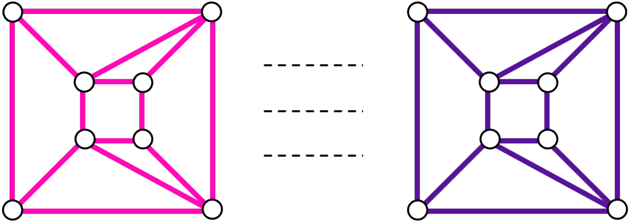

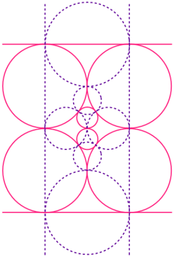

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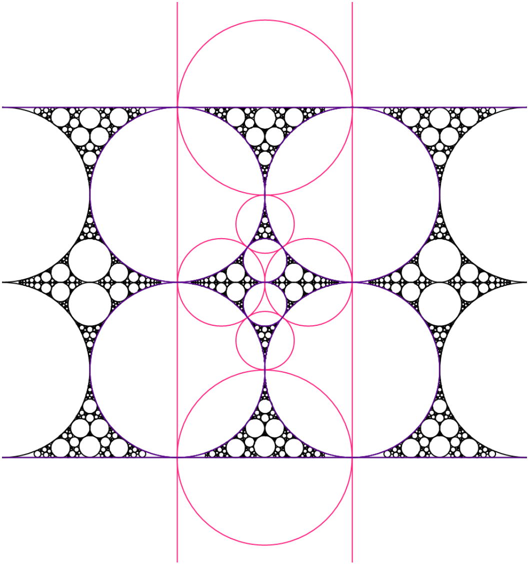


Polyhedra: Methods

- Polyhedron data was generated with **plantri**, a program created by Brinkmann and McKay
- We wrote code in Mathematica using some techniques from Ziegler 2004
- Data is being collected and presented on our website

Polyhedra: Website

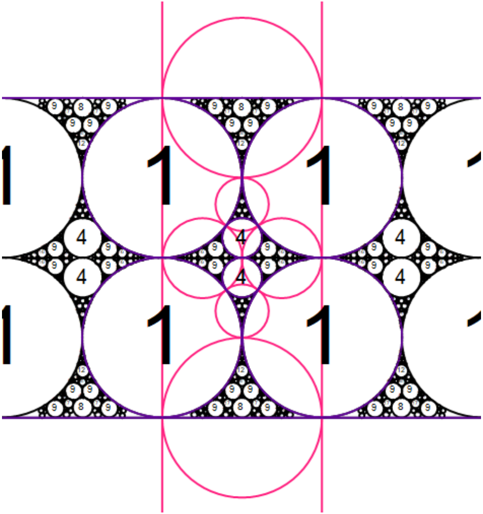
Polyhedral Circle Packings

Click to expand

Polyhedron	Strip Supercluster	Strip Packing	Gramian Matrix	Inversive Coordinates	Bend Matrices	Mathematica File
Tetrahedron 			$\begin{pmatrix} -1 & 1 & 1 & 1 & 0 & 0 & 0 & 2 \\ 1 & -1 & 1 & 1 & 0 & 0 & 2 & 0 \\ 1 & 1 & -1 & 1 & 0 & 2 & 0 & 0 \\ 1 & 1 & 1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 & 1 & -1 & 1 & 1 \\ 0 & 2 & 0 & 0 & 1 & 1 & -1 & 1 \\ 2 & 0 & 0 & 0 & 1 & 1 & 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 4 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 4 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 4 & 1 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 4 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 2 & -1 \end{pmatrix}$ Integral	Code
Square Pyramid 			$\begin{pmatrix} -1 & 1 & 3 & 1 & 1 & 0 & 2\sqrt{2} & 0 & 2\sqrt{2} & 0 \\ 1 & -1 & 1 & 3 & 1 & 0 & 0 & 0 & 2\sqrt{2} & 2\sqrt{2} \\ 3 & 1 & -1 & 1 & 1 & 2\sqrt{2} & 0 & 0 & 0 & 2\sqrt{2} \\ 1 & 3 & 1 & -1 & 1 & 2\sqrt{2} & 2\sqrt{2} & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & -1 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 2\sqrt{2} & 2\sqrt{2} & 0 & -1 & 1 & 1 & 3 & 1 \\ 2\sqrt{2} & 0 & 0 & 2\sqrt{2} & 0 & 1 & -1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 1 & 1 & -1 & 1 & 1 \\ 2\sqrt{2} & 2\sqrt{2} & 0 & 0 & 0 & 3 & 1 & 1 & -1 & 1 \\ 0 & 2\sqrt{2} & 2\sqrt{2} & 0 & 0 & 1 & 3 & 1 & 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 8 & 1 & 2\sqrt{2} & 1 \\ 4 & 2 & 2\sqrt{2} & 1 \\ 0 & 1 & 0 & 1 \\ 4 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 4\sqrt{2} & \sqrt{2} & 3 & 0 \\ 0 & \sqrt{2} & 1 & 0 \\ 2\sqrt{2} & \frac{1}{\sqrt{2}} & 1 & \sqrt{2} \\ 0 & 0 & -1 & 0 \\ 4\sqrt{2} & 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 6 & -3 & 2 & 4 \\ 0 & 6 & -4 & 3 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ Integral	Code

Polyhedra: Findings

Interested in which polyhedra give rise to integral packings



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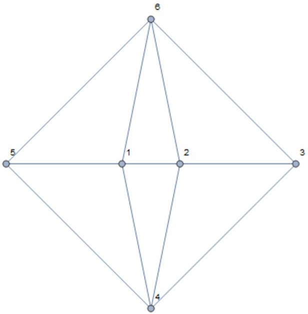
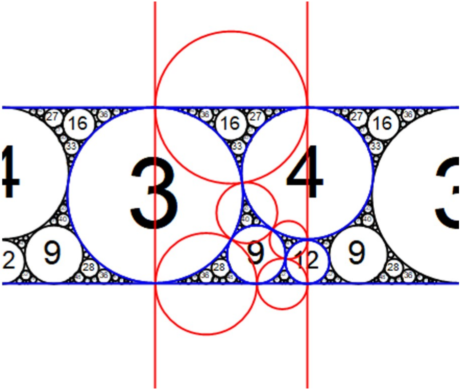
  - Define a **gluing** operation to be a joining along vertices or faces
- Since the tetrahedron, square pyramid, and hexagonal pyramid cannot be decomposed (unglued) into smaller integral polyhedra, they can be considered **seed polyhedra**
- We found a new integral seed polyhedron!

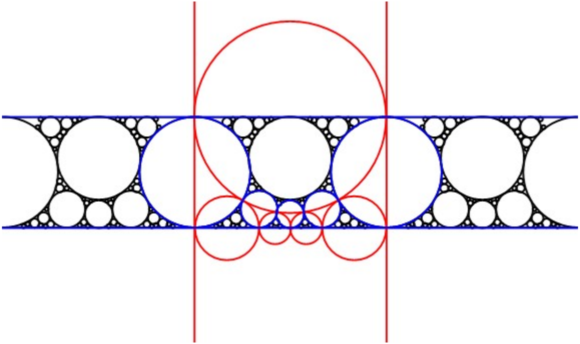
Polyhedra: Findings - 6v7f_2



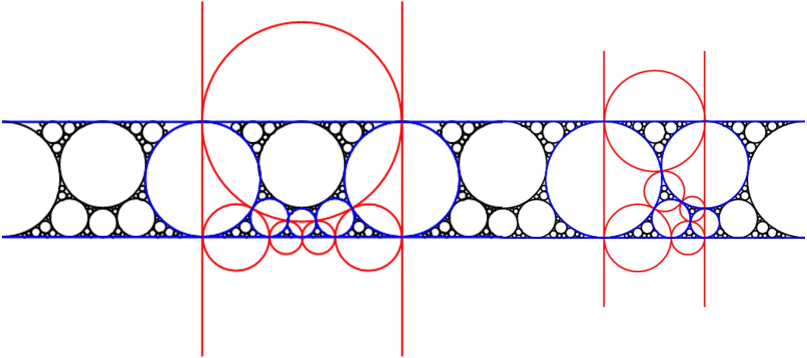


Polyhedra: Findings - 6v7f_2

This is the packing of a hexagonal pyramid; it is in fact the same packing as 6v7f_2.



Polyhedra: Findings - 6v7f_2



Bianchi Groups

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Luigi Bianchi began studying these groups over 100 years ago, in 1892... Before they were ever connected to circle packings!

Bianchi Groups

Sui gruppi di sostituzioni lineari con coefficienti appartenenti a corpi quadratici immaginarî.

Di

Luigi Bianchi a Pisa.

Prefazione.

La presente Memoria tratta dei gruppi di sostituzioni lineari:

$$z' = \frac{\alpha z + \beta}{\gamma z + \delta} \tag{1}$$

sopra una variabile complessa z, i cui coefficienti $\alpha, \beta, \gamma, \delta$ percorrono tutti i numeri *interi* di un *corpo quadratico immaginario* Ω, assoggettati alla sola condizione

$$\alpha\delta - \beta\gamma = 1.\text{*)} \tag{2}$$

Essa è una continuazione del lavoro da me pubblicato nel Vol[e] XXXVIII di questi Annali, ove già è indicata la generalizzazione, che qui trova il suo effettivo svolgimento.**)

Ogni numero intero o frazionario in Ω ha la forma:

$$m + in\sqrt{D}, \tag{3}$$

Bianchi Groups

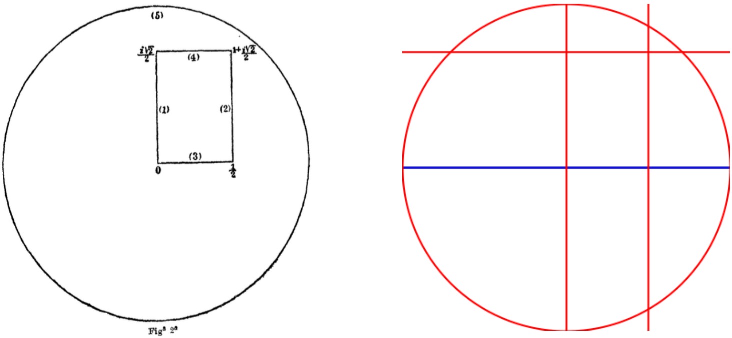

Figure: Bi(2): From 1892 to 2018

Bianchi Groups

Bianchi was interested in exploring which Bianchi groups are *reflective*, meaning finitely generated. 120 years later, Belolipetsky and McLeod conclusively showed that there are a finite number of these reflective Bianchi groups, and enumerated them.

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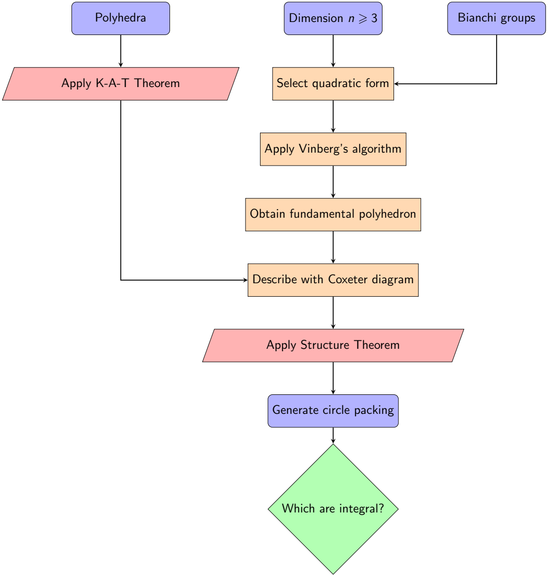
The reflective Bianchi groups can be used to generate circle packings. But how do we go from matrices to circles?

Bianchi Groups



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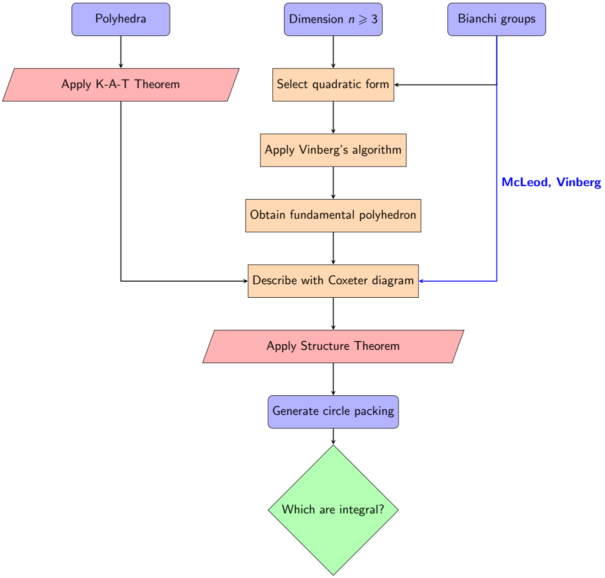

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Polyhedral Circle Packings

Click to expand

Bianchi Group Packings

Click to expand

$$-x_0^2 + \sum_{i=1}^{n} x_i^2$$

Click to expand

$$-2x_0^2 + \sum_{i=1}^{n} x_i^2$$

Click to expand

$$-3x_0^2 + \sum_{i=1}^{n} x_i^2$$

Click to expand

Bianchi Groups

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Polyhedral Circle Packings

Click to expand

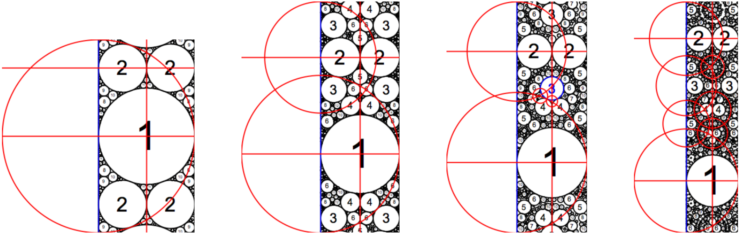
Bianchi Group Packings

Click to expand

Group	Visualization	Coxeter Diagram	Strip Packings	Gramian Matrix	Inversive Coordinates	Bend Matrices	Mathematica File
Bi(1)				$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & -1 \end{pmatrix}$	[illegible]	[illegible]	Code
Bi(2)				$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{2} & 0 & \frac{1}{\sqrt{2}} & -1 \end{pmatrix}$	[illegible]	[illegible]	Code
Bi(3)			None	$\begin{pmatrix} -1 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} & \frac{1}{2} \\ \ldots \end{pmatrix}$	[illegible]	None	Code

Bianchi Groups and Integrality

An interesting property of Bianchi group circle packings is that most of them are *integral*.



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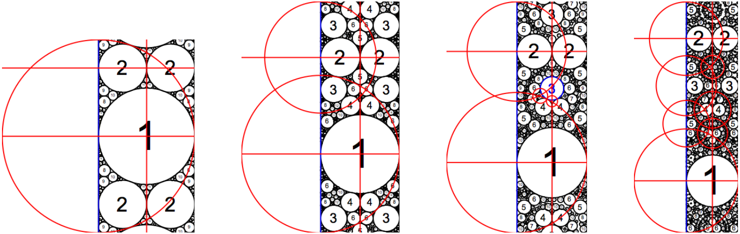

It's very easy to see that a packing is integral empirically just by computing the bends of the packing, but there's actually a way to *prove* integrality more rigorously.

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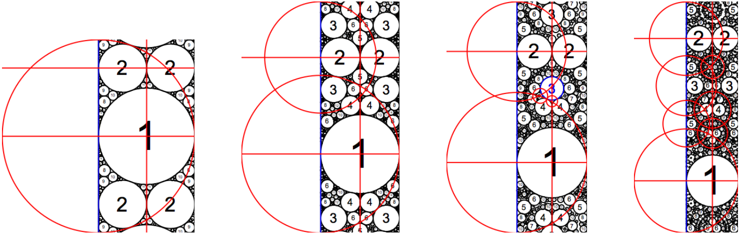

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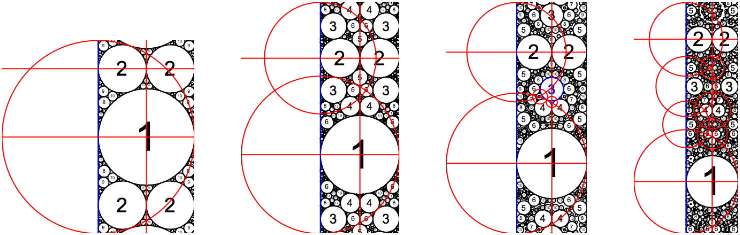


It's very easy to see that a packing is integral empirically just by computing the bends of the packing, but there's actually a way to *prove* integrality more rigorously.

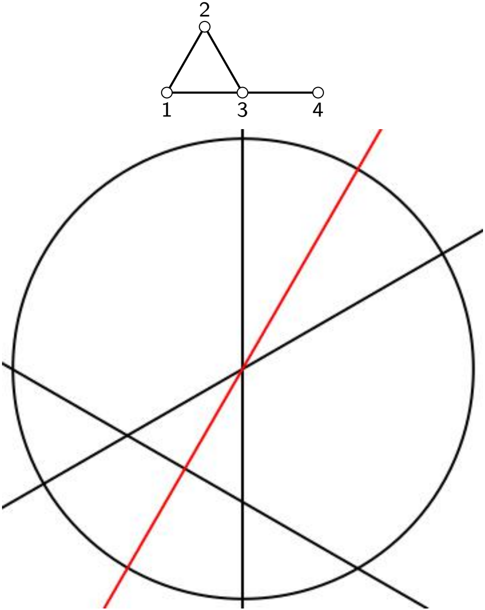
Likewise, there's a way to prove nonintegrality of a packing.

An exciting part of our work this summer is proving integrality and nonintegrality for all known Bianchi packings.

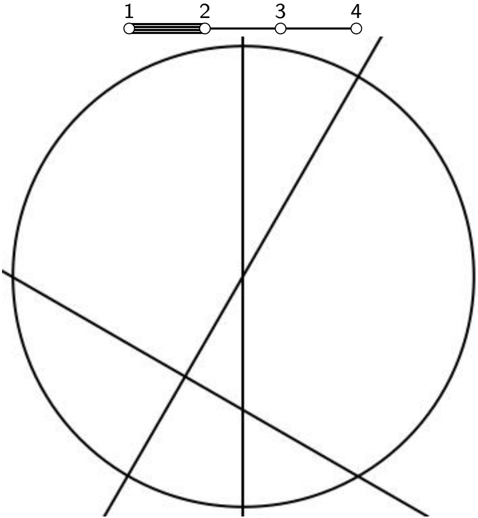
Doubling in $\hat{B}i(3)$

One note, which will also be relevant shortly, is that an insight in Kontorovich & Nakamura's 2017 paper was the observation that what was thought to be the $\hat{B}i(3)$ Coxeter diagram did not actually represent the full group of mirrors:

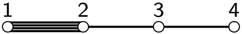
Doubling in $\hat{B}i(3)$

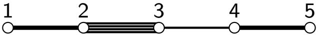


Doubling in $\hat{B}i(3)$



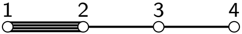
Doubling in $\hat{Bi}(3)$

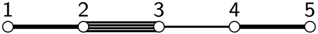
Through a further series of operations, we can transform the diagram 

into the diagram


.

Doubling in $\hat{B}i(3)$

Through a further series of operations, we can transform the diagram 

into the diagram 

. However, this was done less systematically; it primarily derived from looking at the orbit of the original generators acting on themselves until a valid configuration was found that has an isolated cluster.

Questions About Existence of Packings

Now that we've covered polyhedra and Bianchi groups, which give all the crystallographic packings in two dimensions, the natural question is: Do higher dimensional packings exist? What can we say about them?

Questions About Existence of Packings

Now that we've covered polyhedra and Bianchi groups, which give all the crystallographic packings in two dimensions, the natural question is: Do higher dimensional packings exist? What can we say about them? We can answer this by looking at Coxeter diagrams of higher-dimensional configurations, and applying the Structure Theorem, which still holds.

High-Dimensional Packings

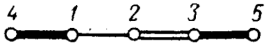
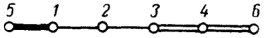
This summer, I worked on putting together the known candidates for these packings, namely 41 potential packings for quadratic forms $-dx_0^2 + \sum_{i=1}^{n} x_i^2$ for $d = 1, 2, 3$.

High-Dimensional Packings

This summer, I worked on putting together the known candidates for these packings, namely 41 potential packings for quadratic forms $-dx_0^2 + \sum_{i=1}^{n} x_i^2$ for $d = 1, 2, 3$. Here is a snapshot of how some appear on our website:

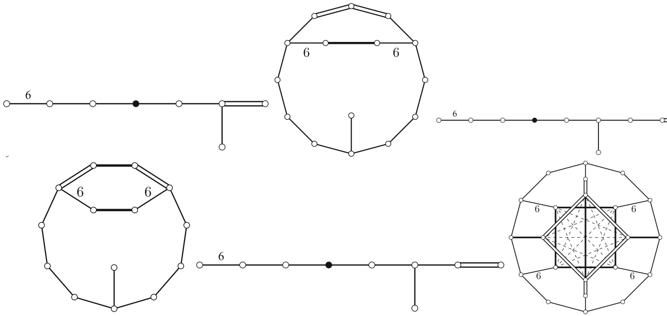
$-2x_0^2 + \sum_{i=1}^{n} x_i^2$

Click to expand

n	Inversive Coordinates	Coxeter diagram	Gram matrix	Packing (for d=2,3)	Bend Matrices	Mathematica File
2	$\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & -1 \\ 1+\sqrt{2} & 1-\sqrt{2} & 0 \end{pmatrix}$		$\begin{pmatrix} -1 & \frac{1}{\sqrt{2}} & 1 \\ \frac{1}{\sqrt{2}} & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$	packing	[illegible]	Code
3	$\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & -1 \\ 1+\sqrt{2} & 1-\sqrt{2} & 0 & 0 \\ 1+\sqrt{2} & \sqrt{2}-1 & 1 & 1 \end{pmatrix}$		$\begin{pmatrix} -1 & \frac{1}{2} & 0 & 1 & 0 \\ \frac{1}{2} & -1 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix}$		[illegible]	Code
4	$\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & -1 \\ 1+\sqrt{2} & 1-\sqrt{2} & 0 & 0 & 0 \\ 1+\frac{1}{\sqrt{2}} & 1-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$		$\begin{pmatrix} -1 & \frac{1}{2} & 0 & 0 & 1 & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -1 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -1 & 0 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & -1 \end{pmatrix}$		[illegible]	Code

High-Dimensional Packings

The techniques discussed here have also allowed us to attack the following diagrams that lack isolated clusters:



Source: John McLeod

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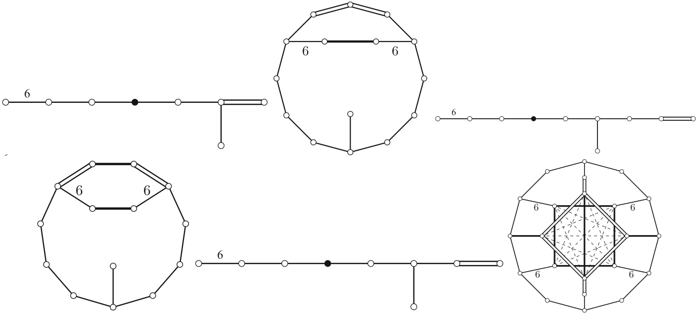
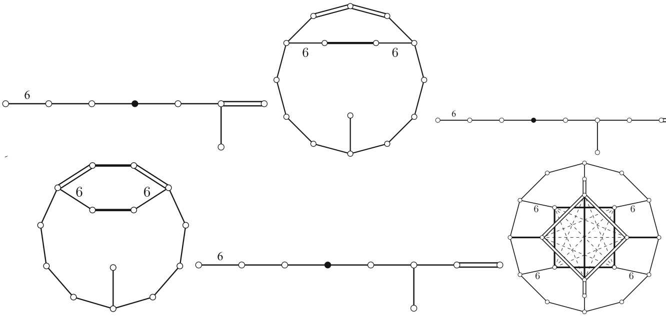


Source: John McLeod

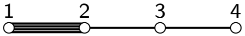
What's something that all of these have in common?

High-Dimensional Packings

The techniques discussed here have also allowed us to attack the following diagrams that lack isolated clusters:

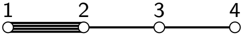


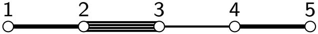
Source: John McLeod

What's something that all of these have in common? They all feature 

as a subdiagram!

High-Dimensional Packings

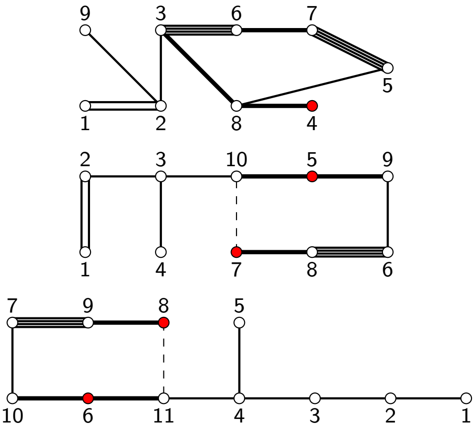
So, if we apply the known transformation for 

into 

followed by a suitable action on the remainder of the nodes in the Coxeter diagram, then hopefully we will obtain a valid diagram representing one such desired subgroup of mirrors.

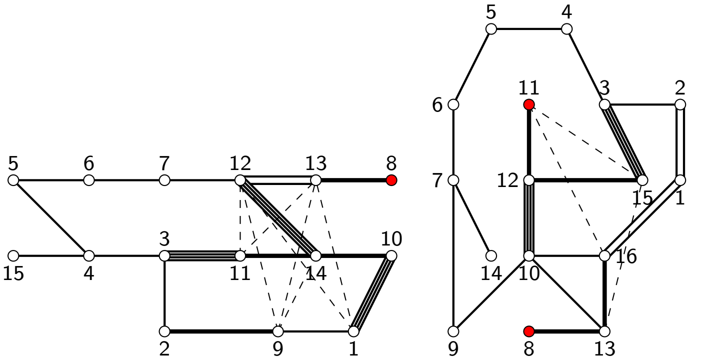
Results

The following Coxeter diagrams were obtained for the $n = 6, 7, 8$ cases of the quadratic form $-3x_0^2 + \sum_{i=1}^{n} x_i^2$:



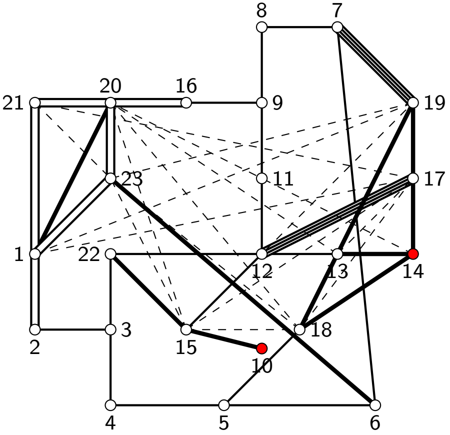
Results

The following is believed to work for $n = 10$, and works for $n = 11$:



Results

Lastly, this behemoth is a diagram for $n = 13$:



References

We are much indebted to the following papers:

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- E. Vinberg, “On Groups of Unit Elements of Certain Quadratic Forms,” 1972.
- G. Ziegler, “Convex Polytopes: Extremal Constructions and f-Vector Shapes,” 2004.

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