YOUNG DIAGRAMS AND PARTIAL FLAG VARIETIES

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1. Introduction

Consider $\{e_1, e_2, \cdots, e_n\}$, a fixed ordered basis of $\mathbb{C}^n$. Recall that the standard flag of $\mathbb{C}^n$ is defined by $E_1 \subset E_2 \subset \cdots \subset E_n$, where $E_k = \text{span}\{e_1, e_2, \cdots, e_k\}$. We now introduce partial flag varieties. Let $m = (m_1, m_2 \cdots, m_k)$, where $0 < m_1 \leq m_2 \leq \cdots \leq m_k < n$. Let $X = Fl(m, n)$, a partial flag variety, then $X = \{(V_{m_1} \subset V_{m_2} \subset \cdots \subset V_{m_k} \subseteq \mathbb{C}^n) \mid dim(V_{m_i}) = m_i\}$. Let $E_m = (E_{m_1} \subset E_{m_2} \subset \cdots \subset E_{m_k}) \in X$ and let P denote the stabilizer of E_m, i.e. $\{g \in GL(n) \mid g.E_m = E_m\}$. Let $m_0 = 0$ and $m_{k+1} = n$, then P is the group of invertible block upper triangular matrices, where the dimension of the i^{th} block is $m_{i+1} - m_i$. The following are other key subgroups of $GL(n)$ that we will consider: T is the torus and the set of invertible diagonal matrices, B is the Borel subgroup and is the set of invertible upper triangular matrices. We have the following relation among these subgroups: $T \subseteq B \subseteq P \subseteq GL(n)$. We will also only consider Weyl groups of type A so $W = S_n$.

2. Exercises

Exercise 1. We shall show that $X^T = \{w.E_m \mid w \in S_n\}$.

It is easy to see that $w.E_m \in X^T$. Now to show the other direction, consider $t \in T$ and write t as

$$\begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

Let $M \in X$ and write M as $\begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}$ where v_i is a $n \times 1$ vector. Then in order for M to be in X, we must have that the span of the first m_1 vectors after multiplying by scalars from t must be the same as E_{m_1} so it follows that there can only be m_1 nonzero rows among these first m_1 vectors because we allow t to vary. Thus using column operations, we can get that these m_1 vectors only have m_1 nonzero entries combined and each is in a different row. We continue this process for E_{m_2} all the way to E_{m_k} and $\mathbb{C}^n$ from which we can see that M has the form $w.E_m$ for some $w \in S_n$.

Normalizer. Define $N_G(T) = \{g \in GL(n) \mid gTg^{-1} = T\}$. One can show that $N_G(T) = \{$all permutations matrices with arbitrary nonzero numbers in the 1's places$\} = S_nT$. We now show some relations between $N_G(T)$ and W.

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Exercise 2. $W = S_n = N_G(T)/T$ and $W_P = S_n \cap P = N_P(T)/T$ and furthermore $W_P \leqslant W$.

First proof that$N_G(T) = S_nT$. Let $M \in N_G(T)$ and let T' be a diagonal matrix such that $T_{ii} = i$, then $MT'M^{-1} = T^*$ is also a diagonal matrix. Since conjugation preserves spectrum, it must be that $T^*_{ii} = \sigma(i)$ for some permutation σ. Since $MT' = T^*M$, $S_{ij}j = \sigma(i)S_{ij}, S_{ij}(j - \sigma(i)) = 0$. So $S_ij = 0$ for all $\sigma(i) \neq j$. Each row has only one non-zero entry. So M must be of the form S_nT.

Now since $N_G(T) = S_nT$, the homomorphism $\phi N_G(T) \to W$ where $\phi(M) = w$ with $M = wT'$ for some permutation w and diagonal matrix T'. w is the identity permutation if and only if $M \in T$. So the kernel of ϕ is T. Therefore $N_G(T)/T \cong W$. Therefore we have our desired isomorphism and a similar argument works for W_P.

Note that W_P is made up of the permutations in S_n that fit the shape of P. It is clear from the definition of P that the identity element is in W_P. If you consider the transpositions that generate all the possible permutations in one specific block of P, then it is easy to see that $W_P \leqslant W$ since these transpositions generate W_P.

Exercise 3. We will show that for $w \in W$ there exists a unique permutation $u \in wW_P$ of minimal length where length of $u = l(u) = \#\{(i < j) \mid u(i) > u(j)\}$.

We provide the following algorithm for constructing u. Start with w and multiply w by $s \in W_P$ only if $l(ws) < l(w)$. Do this for all elements $s \in W_P$ so that at the end, we have $u = ws_{a_1}s_{a_2}\cdots s_{a_k}$. The claim is that this u is of minimal length. For sake of contradiction, suppose we have $v \in W_P$ such that $l(v) < l(u)$. Then this means that there exists $s \in W_P$ such that $vs = u$ so $v = us^{-1}$. Therefore $l(us^{-1}) < l(u)$, however this is not possible by construction of u. u is also unique. Suppose u_1 and u_2 both have minimal length. Then $u_1 = u_2w_2$ and $u_2 = u_1w_1$ for w_1,$w_2 \in W_P$. Hence $l(u_1) \leq l(u_1w_1) = l(u_2)$ and $l(u_2) \leq l(u_2w_2) = l(u_1)$, by definition of u_1 and u_2 being elements of minimal length in wW_P. We also have $l(u_1) = l(u_2)$ so it follows that $w_1 = w_2 = e$ and therefore, $u_1 = u_2$.

Grassmannian. Let $X = Gr(m, n)$. We will view this Grassmannian as a partial flag variety. P consists of invertible block upper triangular matrices with two blocks. The block in the top left has size m and the block in the bottom right has size $n-m$. We now describe W_P in terms of its generators. The generators are $\{(1i) \mid 2 \leq i \leq m\} \cup \{((m+1)j) \mid m+2 \leq j \leq n\}$. Let $W^P \subseteq W$ be the set of all such u described in exercise 3. In X, W^P is the set of all permutations that send $(12\ldots m)$ to m numbers that are ordered from lowest to highest so there are $\binom{n}{m}$ elements in W^P. Therefore we can discuss a bijection between young diagrams and W^P. Note that we already have a bijection between young diagrams and X^T since X^T corresponds to Schubert symbols, which are used to construct a young diagram.

Exercise 4. We shall find a bijection between X^T and W^P and between young diagrams and X^T.

We will show a bijection between W/W_P and X^T since W/W_P is essentially equivalent to W^P. Let $w.E_m \in X^T$ be a arbitrary element. We prove that $\phi(w.E_m) = wW_p$ is a bijection. By exercise three, we know there exists a unique minimal length u for wW_p. since $uW_p = wW_p$, $w = uv$ for some $v \in W_p$. Now by definition of stabilizer $v.Em = Em$ for all $v \in W_p$. So $\phi(w.E_m) = \phi(uv.E_m) == \phi(u.E_m) = uW_p$. Since such a u exist and is unique. ϕ is a bijection. A bijection with young diagrams follows through bijection between young diagrams and X^T.

Note that by this bijection, $l(u) = |\lambda|$. Cool! In fact, we can say more about the connection between u and λ. For $1 \leq i \leq m$, the number of boxes in row i corresponds with the number of inversions with i, i.e. the number of boxes $= \#\{j \mid j > i \text{ and } u(i) > u(j)\}$. Therefore, u will completely determine λ.