SCHUBERT VARIETIES AND CURVE NEIGHBORHOODS

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1. Introduction

We start by defining key terms. Let $X = Gr(m, n)$ denote a fixed Grassmannian. Consider $\{e_1, e_2, \cdots, e_n\}$, a fixed ordered basis of $\mathbb{C}^n$. The standard flag of $\mathbb{C}^n$ is defined by $E_1 \subset E_2 \subset \cdots \subset E_n$, where $E_k = \text{span}\{e_1, e_2, \cdots, e_k\}$. Let X_I be the associated Schubert Variety for some Schubert symbol I. It has been shown that $X_I = \{V \in X \mid dim(V \cap E_k) \geq \#(I \cap [1, k]\}$. Let Ω be a closed subset of X. $\Gamma_d(\Omega)$ is the curve neighborhood of Ω, which is defined to be the union of all curves of degree d that intersect Ω.

2. Exercises

Exercise 1. We shall show that $\Gamma_1(E_m) = X_I$, where $I = \{2, 3, \cdots, m, n\}$.

Suppose $V \in \Gamma_1(E_m)$. Then this means that there exists a line L in X such that V and E_m are both contained in line L. Recall that a line in the Grassmannian is defined by two subspaces of $\mathbb{C}^n$, A and B, where $dimA = m - 1$ and $dimB = m + 1$, i.e. $L = \{V \in X \mid A \subseteq V \subset B\}$. Therefore we have the following set of relations: $A \subseteq V \subset B$ and $A \subseteq E_m \subset B$. This implies that

$$dim(V \cap E_m) \geq m - 1. \tag{1}$$

It can be shown that (1) further implies that for $1 \leq k \leq m - 1$, $dim(V \cap E_k) \geq k - 1$. Note that it is always true that $dim(V \cap E_n) = m$, so using (1), we have that for $m + 1 \leq k \leq n$, $dim(V \cap E_k) \geq m - 1$. Hence, if we put all these inequalities together, we have that $V \in X_I$, where $I = \{2, 3, \cdots, m, n\}$. Now suppose $V \in X_I$, then we must certainly have that $dim(V \cap E_m) \geq m - 1$. This is the same as saying there exists an $m - 1$-dimensional subspace A and and $m + 1$-dimensional subspace B such that $A \subseteq V \subset B$ and $A \subseteq E_m \subset B$, which means that there is a line $L \in X$ such that L connects V and E_m, so $V \in \Gamma_1(E_m)$.

Exercise 2. Let $S \subseteq Gr(m, n)$, proof that $\Gamma_d(S) = \Gamma_1(\Gamma_{d-1}(S))$.

$\Gamma_d(S) \subseteq \Gamma_1(\Gamma_{d-1}(S))$:
Let $V \in S$ and $U \in \Gamma_d(S)$. By previous exercises we know that there's a chain of d intersecting lines that go from V to U. Let A be the intersection point that is colinear with U, then a chain of $d - 1$ lines go from V to A, so $A \in \Gamma_{d-1}(S)$. Since there is a line from A to U, $U \in \Gamma_1(A)$. Since $A \subseteq \Gamma_{d-1}(S)$, $\Gamma_1(A) \subseteq \Gamma_1(\Gamma_{d-1}(S))$, we have $U \in \Gamma_1(\Gamma_{d-1}(S))$.

$\Gamma_1(\Gamma_{d-1}(S)) \subseteq \Gamma_d(S)$:
Let $V \in S$ and $U \in \Gamma_1(\Gamma_{d-1}(S))$. Since $U \in \Gamma_1(\Gamma_{d-1}(S))$, there must be some $A \in \Gamma_{d-1}(S)$ that V is colinear to. Since $A \in \Gamma_{d-1}(S)$, there's a chain of $d - 1$ intersecting lines that go from V to A. By appending the line from a to U to the path, we have a chain of d intersecting lines that go from V to U, so $U \in \Gamma_d(S)$.

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Exercise 3. (The previous exercises have already shown that $X_I = Z_I$.) Proof that $\Gamma_d(X_I) = Z_J$, where $J \subseteq [1,n]$ is defined by taking I, and swapping the d smallest elements in I with the d largest elements in $[1,n] - I$.

First show that this holds for $d = 1$.

$\Gamma_d(X_I) \subseteq Z_J$:

Let $V \in Z_I$, so $dim(V \cap E_k) \geq \#(I \cap [1,k])$ for all k. Let $U \in \Gamma_1(Z_I)$, then there exists some line through V and U. So $dist(V,U) = 1$, by previous exercise we have $dim(V+U) = m+1$ thus $dim(V \cap U) = m-1$. Let $A = V \cap U$, then V, U can be expressed as $V = A + V', U = A + U'$ where V', U' has dimension 1. Let i be the smallest element of I and j the largest element of $[1,n] - I$. By construction of J, we see that for all k,

$$\#(J \cap [1,k]) = \begin{cases} 0 \text{ for } k < i \\ \#(I \cap [1,k]) - 1 \text{ for } i \leq k < j \\ \#(I \cap [1,k]) \text{ for } j < k \leq n \end{cases}$$

For $k < i$, $dim(U \cap E_k) \geq 0$ is obvious.

For $i \leq k < j$, $dim(E_k \cap U) = dim(E_k \cap (A + U')) \geq dim(E_k \cap A) \geq dim(E_k \cap (A + V')) - 1 = dim(E_k \cap V) - 1 \geq \#(I \cap [1,k]) - 1 = \#(J \cap [1,k])$. We know $dim(E_k \cap A) \geq dim(E_k \cap (A + V')) - 1$ since V' has dimension 1, contributing at most 1 to the total dimension.

For $j < k \leq n$, assume $dim(E_k \cap U) < \#(I \cap [1,k])$, since j is the largest element not in $[1,n] - I$ for all positive integer c where $k + c < n$, $\#(I \cap [1,k+c]) = \#(I \cap [1,k]) + c$. Now let $k + c = n$. $dim(E_n \cap U) = dim(E_{k+c} \cap U) \leq dim(E_k \cap U) + c < \#(I \cap [1,k]) + c = \#(I \cap [1,k+c]) = \#(I \cap [1,n]) = m$. We have $dim(E_n \cap U) < m$, a contradiction. So $dim(E_k \cap U) \geq \#(I \cap [1,k])$.

We see that $dim(E_k \cap U) \geq \#(J \cap [1,k])$ for all cases of k, so $U \in Z_J$. Thus $\Gamma_d(X_I) \subseteq Z_J$.

$Z_J \subseteq \Gamma_d(X_I)$:

Let $U \subseteq Z_J$. For the non-trivial case assume $U \notin X_I$. Take the reduced column echelon form of the matrix that spans U as the basis B_U for U. Since it is in RCEF form, for m of the standard basis e_q, there is exactly one basis vector of the form $e_q + c_r e_r + c_s e_s ...$, where q is the largest index in the linear combination, which shall be denoted as u_q. Let i be the smallest element of I and j the largest element of $[1,n] - I$. Let j be the t^{th} largest element in J. Let q be the t^{th} largest index of all the basis vector indexes in B_U. Notice for U to be in Z_J we have $q \leq j$. Replace u_q with the smallest standard basis that is linearly independent from $B_U - u_q$. We call the span of this new basis U'.

For $k < i$, $dim(U' \cap E_k) \geq 0$ is obvious.

For $i \leq k < j$, if $q < i$ then $dim(U' \cap E_k) \geq \#(J \cap [1,k]) + 1 = \#(I \cap [1,k])$. If $q \geq i$ then the same is true since there must be some vector taking up the spaces less then i.

For $j < k \leq n$, $dim(U' \cap E_k) \geq \#(J \cap [1,k]) = \#(I \cap [1,k])$.

We see that $U' \in X_I$. But then we can define a line from U' to U, setting $A = U \cap U'$ and $B = U + U'$, so $U \in \Gamma_d(X_I)$. $Z_J \subseteq \Gamma_d(X_I)$:

Since this holds for $d = 1$, with exercise 2, we can apply Γ recursively to get the general case for any d.