SCHUBERT CELLS AND YOUNG DIAGRAMS

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1. Introduction

We start by defining key terms. Let $X = Gr(m, n)$ denote a fixed Grassmannian. Consider $\{e_1, e_2, \cdots, e_n\}$, a fixed ordered basis of $\mathbb{C}^n$. The standard flag of $\mathbb{C}^n$ is defined by $E_1 \subset E_2 \subset \cdots \subset E_n$, where $E_k = \text{span}\{e_1, e_2, \cdots, e_k\}$. Let $B \subset GL(n)$ be the Borel subgroup. Let $B.V_I$ be the associated Schubert cell and X_I be the associated Schubert Variety for some Schubert symbol I where $I = \{i_1, i_2, \cdots, i_m\}$. Let λ denote the Young diagram for I, which is constructed as follows: Given an $(n-m)$ x m matrix, you move up in the i^{th} step if $i \in I$, otherwise you move right. The resulting Young diagram is obtained by taking all the boxes in top left part of the matrix.

2. Exercises

Exercise 1. We first obtain a formula for $|\lambda|$, the number of boxes in a Young diagram given its Schubert symbol I.

Let I be the Schubert symbol for λ . As we form the Young diagram through a series of up and right moves on a $(n-m) \times m$ matrix determined by I, we see that every time we move up, all the boxes to the left in that row are included in λ. Now suppose we we move up at step i_j, then this means out of a total of i_j moves, j of them are moves up and $i_j - j$ are moves to the right. Therefore we see that the number of boxes to the left of step i_j is just $i_j - j$ so it follows that $|\lambda|$ is

$$\sum_{j=1}^{m} i_j - j. \tag{1}$$

Exercise 2. We now show that a Schubert cell $B.V_I \cong \mathbb{C}^{|\lambda|}$

We first begin by showing the number of free entries in $B.V_I$ is the same as $|\lambda|$. Consider $b \in B$ and write b as

$$\begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} & \dots & b_{1,n} \\ 0 & b_{2,2} & b_{2,3} & \dots & b_{2,n} \\ 0 & 0 & b_{3,3} & \dots & b_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & b_{n,n} \end{bmatrix}$$

Since $V_I = \text{span}\{e_{i_1}, e_{i_2}, \cdots, e_{i_m}\}$, we can represent V_I as an $n \times m$ matrix, where the j^{th} column is e_{i_j}. Denote the matrix $b.V_I$ as M; this is an $n \times m$ matrix whose j^{th} column looks like the following:

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$$\begin{bmatrix} b_{1,i_j} \\ \vdots \\ b_{i_j,i_j} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Now we can scale the matrix M by column operations so that entry $M_{i_j,j}$ is 1, i.e. the j^{th} column becomes:

$$\begin{bmatrix} \widetilde{b}_{1,i_j} \\ \vdots \\ \widetilde{b}_{i_{j-1},i_j} \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

We can perform more column operations so that for some entry $M_{m,n}$, if $m = i_j$ and $n > j$, then $M_{m,n} = 0$. Therefore for a given column j, the number of free entries is $i_j - j$ so the total number of free entries in $B.V_I$ is precisely the quantity given by (1).

Since $V_I = V_\lambda \in X$, we have a bijection between $B.V_\lambda$ and $\mathbb{C}^{|\lambda|}$ as desired.