Spherical Harmonics

Mathematically Constructing a 2-d nonlinear σ-model

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Outline

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   - S^2
   - $V^{\otimes k}$
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2. Differential Operators
   - Beltrami-Laplace Operator
   - Motivation

Geometry
S^2

- S^2 is the two dimensional unit sphere that lies in $\mathbb{R}^3$.
- Every point on the sphere has a tangent space denoted T_pS^2, which is isomorphic to $\mathbb{R}^2$.
- A vector in T_pS^2 locally defines a differential operator acting on $C^\infty(S^2)$.
- $\bar{v} \cdot f$ is the directional derivative of f along $\bar{v}$.

Geometry

$V^{\otimes k}$

- Let V denote $T_P S^2$.Then we can consider V tensored with itself k times.
- The group SO_2 acts on $V^{\otimes k}$ in the natural way.
- Consider $(V^{\otimes k})^{SO_2} = \{x \in V^{\otimes k} \mid gx = x \forall g \in SO_2\}$.
- Problem that was recently solved by Lehrer and Zhang in December 2016 was to give a basis for the above invariant space.

Geometry

$V^{\otimes k}$

- The reason we need to consider the tensor of V is because otherwise the invariant space will be empty.
- Consider $V^{\otimes 2}$. If we write $V = \mathbb{C}e_+ \oplus \mathbb{C}e_-$, then if we act by some element $g \in SO_2$, then $g \cdot e_+ \otimes e_- = e^{i\theta}e_+ \otimes e^{-i\theta}e_- = e_+ \otimes e_-$.
- Need number of pluses to match number of minuses.

Differential Operators

Beltrami-Laplace Operator

- Every tensor in $(V^{\otimes k})^{SO_2}$ defines globally a differential operator on $C^\infty(S^2)$.
- The Beltrami-Laplace operator Δ sits inside some $(V^{\otimes k})^{SO_2}$
- The eigenfunctions f of Δ correspond to Legendre polynomials.

Differential Operators
Motivation

- Would like to construct a 2-d nonlinear σ-model (a type of quantum field theory)
- Accomplish this by generalizing the work of Lehrer and Zhang and consider the space $\coprod(V^{\otimes k})^{SO_2}$

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References

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