Schubert Calculus

Curve Neighborhoods of a Point

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Geometry

Euclidean and Projective

- Euclidean geometry is the traditional geometry learned in high school that is characterized by the parallel postulate.
- Projective geometry is different from Euclidean geometry in that we allow parallel lines to intersect.
  - Think of standing in the middle of train tracks and looking into the horizon.
  - Let V be a vector space. Then the projective space $P(V)$ of V is the set of 1-dimensional subspaces of V.
  - $\mathbb{P}^n$ is the set of all lines through the origin in $\mathbb{C}^{n+1}$.

Geometry

Grassmannian

- $X = Gr(m, n)$ is the set of m-dimensional subspaces in a vector space V of dimension n.
- Fix a basis of $\mathbb{C}^n = \{e_1, e_2, \ldots, e_n\}$. If $I = \{1, 2, \ldots, m\}$, then $V_I = span\{e_1, e_2, \ldots, e_m\}$ is a point in X.
- Actions on the Grassmannian
  - $G = GL(n, \mathbb{C})$ is the set of all invertible matrices.
  - B is the set of invertible upper triangular matrices.
  - T is the set of invertible diagonal matrices.
  - Note $T \subset B \subset G$

Curve Neighborhoods

Schubert Varieties

- For some index I, a Schubert Cell is the B orbit of some V_I.
- A Schubert Variety is the closure of a Schubert Cell.
- Schubert cells and Schubert Varieties are uniquely identified by their index I.

Curve Neighborhoods

Schubert Varieties

- Let $X = Gr(m, n)$ and Ω be a closed subset of X. Then the degree d curve neighborhood of Ω, written $\Gamma_d(\Omega)$, is the closure of the union of all curves of degree d that meet Ω at a point.

Theorem

The curve neighborhood of a Schubert Variety is another Schubert Variety.

Curve Neighborhoods

Young Diagrams and Weyl Group

- We can keep track of Schubert Varieties by Young Diagrams and elements of the Weyl Group.
- Take a grid that is $m \times n - m$, then an index I corresponds to a Young diagram λ by the following rule: if $i \in I$, then the i^{th} step of your path is up, otherwise you step to the right.
- Given a Weyl group element w, w acts on the numbers $1, 2, \ldots, m$, so w corresponds to some index J. Also the number of inversions i determine the number of boxes at row i. Furthermore the the length of w is the length of λ.
- We can also obtain the Young diagram of a curve neighborhood by shifted the border of λ by d units.

Curve Neighborhoods

Young Diagrams and Weyl Group

Table: Schubert Varieties of $Gr(2,4)$

Young Diagram	Index	Weyl Group (type A)
$\varnothing$	$\{1,2\}$	e
	$\{1,3\}$	(23)
	$\{1,4\}$	(243)
	$\{2,3\}$	(123)
	$\{2,4\}$	(1243)
	$\{3,4\}$	$(13)(24)$

Flag Varieties

Full Flags and Partial Flags

- The standard flag of $\mathbb{C}^n$ is defined by $E_1 \subset E_2 \subset \cdots \subset E_n$, where $E_k = \text{span}\{e_1, e_2, \cdots, e_k\}$, which is an example of a full flag.
- Let $m = (m_1, m_2 \cdots, m_k)$, where $0 < m_1 \leq m_2 \leq \cdots \leq m_k < n$. Let $X = Fl(m, n)$, a partial flag variety, then $X = \{(V_{m_1} \subset V_{m_2} \subset \cdots \subset V_{m_k} \subseteq \mathbb{C}^n) \mid dim(V_{m_i}) = m_i\}$.
- Let $E_m = (E_{m_1} \subset E_{m_2} \subset \cdots \subset E_{m_k}) \in X$ and let P denote the stabilizer of E_m.
- Note $T \subseteq B \subseteq P \subseteq GL(n)$.

Conjecture

- It is useful to look at Schubert Varieties in terms of their Weyl group element since the Weyl group is related to root systems, and coroots (a dual of roots) correspond to a degree d.
- We can determine the curve neighborhoods of a point by looking at these Weyl group elements and their associated roots and coroots.
- There are special types of roots called P-cosmall determined by our group P that allow us to compute these curve neighborhoods.

Conjecture

Conjecture

Assume that R is simply laced and let $\alpha \in R^+ \setminus R_P^+$. The α is P-cosmall if and only if $z_{\alpha^\vee}^P W_P = s_\alpha W_P$.

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