Note

Geometric drawings of $K_n$ with few crossings

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Abstract

We give a new upper bound for the rectilinear crossing number $\overline{cr}(n)$ of the complete geometric graph $K_n$. We prove that $\overline{cr}(n) \leq 0.380559 \binom{n}{4} + \Theta(n^3)$ by means of a new construction based on an iterative duplication strategy starting with a set having a certain structure of halving lines.

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1. Introduction

The crossing number $cr(G)$ of a simple graph $G$ is the minimum number of edge crossings in any drawing of $G$ in the plane, where each edge is a simple curve. The rectilinear crossing number $\overline{cr}(G)$ is the minimum number of edge crossings in any geometric drawing of $G$, that is a drawing of $G$ in the plane where the vertices are points in general position and the edges are straight segments. The crossing numbers have many applications to Discrete Geometry and Computer Science, see, for example, [9] and [10].

In this paper we contribute to the problem of determining $\overline{cr}(K_n)$, where $K_n$ denotes the complete graph on $n$ vertices. Specifically, we construct geometric drawings of $K_n$ with a small number of crossings. For simplicity we write $\overline{cr}(n) = \overline{cr}(K_n)$. We note that a geometric drawing of $K_n$ is determined by the location of its vertices and two edges cross each other if and only if the quadrilateral determined by their vertices is convex. Thus our drawings also provide constructions of $n$-element point sets with small number of convex quadrilaterals determined by the $n$ points. The problem of finding the asymptotic behavior of $\overline{cr}(n)$ is also important because of its
close relation to Sylvester’s four point problem [13]: Determine the probability that four points selected uniformly at random on a given domain, form a convex quadrilateral. It is easy to see that \( \{ \ell(n)/\binom{n}{4} \} \) is a non-decreasing sequence bounded above by 1. Thus \( v = \lim_{n \to \infty} \ell(n)/\binom{n}{4} \) exists, and Scheinerman and Wilf [12] proved that \( v \) is the infimum, over all open sets \( R \) with finite area, of the probability that four randomly chosen points in \( R \) are in convex position.

When this problem was first investigated, the best lower bounds were obtained by using an averaging argument over subsets whose crossing numbers were known. For example, it is well known that \( \ell(5) = 1 \), so by counting the crossings generated by every subset of size 5 it is easy to get \( \ell(n) \geq (1/5)\binom{n}{4} \). Wagner [14] was the first to use a different approach, he proved \( v \geq 0.3288 \). Then the authors [1] and independently Lovász et al. [8], used allowable sequences to prove \( v \geq 3/8 = 0.375 \). Lovász et al. [8] managed to even improve this \( v \) by \( 10^{-5} \), and Balogh and Salazar [5] refined this technique even further to obtain the currently best bound of \( v \geq 0.37533 \).

The history of the improvements on the upper bounds is as follows: In the early seventies Jensen [7] and Singer [11] obtained \( v \leq 7/18 < 0.3888 \), and \( v \leq 5/13 < 0.38462 \), respectively. Much later Brodsky et al. [6] constructed point sets with nested non-concentric triangles yielding \( v \leq 6467/16,848 < 0.383844 \). Then Aichholzer et al. [3] devised a lens-replacement construction depending on a suitable initial set. They obtained \( v \leq 0.380739 \) and later on, Aichholzer and Krasser found a better initial set [4], which gave the previously known best bound of \( v \leq 0.38058 \).

In this paper we prove the following theorem.

**Theorem 1.** \( \ell(n) \leq \frac{20.969}{18.750} \binom{n}{4} + \Theta(n^3) < (0.380559)\binom{n}{4} + \Theta(n^3) \).

This theorem is based on the following stronger result which may improve the upper bound in the future. To accomplish this we would only need a point-set \( P \) having a halving-line matching (defined next) and with a better \( \binom{n}{4} \) coefficient.

**Theorem 2.** If \( P \) is a \( N \)-element point set in general position, with \( N \) even, and \( P \) has a halving-line matching; then

\[
\ell(n) \leq \left( \frac{24cr(P) + 3N^3 - 7N^2 + (30/7)N}{N^4} \right)\binom{n}{4} + \Theta(n^3).
\]

Let \( P = \{ p_1, p_2, \ldots, p_N \} \) be a general position point set in the plane. We define a halving line of \( P \) as a line passing through two points in \( P \) and leaving the same number of points of \( P \) on either side of the line. According to this, \( N \) needs to be even for \( P \) to have a halving line.

Consider a bipartite graph \( G = (P, H) \) where \( H \) is the set of halving lines of \( P \) and \( p \in P \) is adjacent to \( l \in H \) if \( p \) is on the line \( l \). If there is a matching for \( P \) in the graph \( G \) we say that this matching is a halving-line matching of \( P \). Note that such a matching induces a function on \( P \), \( p_i \mapsto p_{f(i)} \), such that \( p_ip_{f(i)} \) is a halving line of \( P \) and if \( i \neq j \) then \( p_ip_{f(i)}, p_jp_{f(j)} \) are different halving lines.

All known optimal drawings of \( K_n \) with \( n \) even 6 ≤ \( n \) ≤ 16 have a halving-line matching. The same is true for all the best known drawings with 18 ≤ \( n \) ≤ 48 reported in [2]. The only exception seems to be \( P_4 \), the set consisting of a triangle with a point inside. This set achieves \( \ell(4) = 0 \) but it only has three halving lines, so a halving-line matching is impossible.
The previously best general constructions, obtained by Aichholzer et al. [3], used a lensreplacement construction yielding
\[
\bar{cr}(n) \leq \left( \frac{24cr(P) + 3N^3 - 7N^2 + 6N}{N^4} \right) \left( \frac{n}{4} \right) + \Theta(n^3)
\]
for any initial point set \( P \) with \( N \) elements, \( N \) even. This bound requires the same number of points in each of the lenses. They further refined their method by considering lenses of different sizes. Unfortunately this post-optimization depends heavily on the structure of the initial set and it seems that the improved constant cannot be verified other than by computer calculations. In [4], Aichholzer and Krasser use a set of 54 points with 115,999 crossings which gives \( \bar{cr}(n) \leq (0.380601)^n + \Theta(n^3) \) using (1) and they state the bound \( \bar{cr}(n) \leq (0.38058)^n + \Theta(n^3) \) after the post-optimization process (the details about the lens sizes are not available in [4]). Note that our construction from Theorem 2 always gives a better bound compared to (1) as long as the starting set \( P \) has a halving-line matching.

2. The construction

The construction is based on the following lemma.

**Lemma 3.** If \( P \) is a \( N \)-element point set, \( N \) even, and \( P \) has a halving-line matching; then there is a point set \( Q = Q(P) \) in general position, \( |Q| = 2N \), \( Q \) also has a halving-line matching, and \( \bar{cr}(Q) = 16\bar{cr}(P) + (N/2)(2N^2 - 7N + 5) \).

**Proof.** \( Q \) is constructed as follows. Each point \( p_i \in P \) will be replaced by a pair of points \( q_{i1} \) and \( q_{i2} \). Using the function \( f \) induced by the halving-line matching we define
\[
q_{i1} = p_i + \epsilon \frac{p_f(i) - p_i}{\| p_f(i) - p_i \|} \quad \text{and} \quad q_{i2} = p_i - \epsilon \frac{p_f(i) - p_i}{\| p_f(i) - p_i \|},
\]
where \( \epsilon \) is small enough so that all points \( q_{j1}, q_{j2} \) coming from a point \( p_j \) located to the left (right) of \( \overrightarrow{p_i p_f(i)} \); are also located to the left (right) of the lines \( q_{i1} q_f(i)1 \) and \( q_{i1} q_f(i)2 \). In other words, the cone with vertex \( q_{i1} \) spanned by the segment \( q_f(i)1 q_f(i)2 \) does not contain any of the points \( q_{jx} \) with \( j \neq i, f(i) \). One way to find such an \( \epsilon \) is the following: suppose that \( P \) is contained in a disk with diameter \( D \), since \( P \) is in general position there is \( \delta > 0 \) so that the strips of width \( \delta \) centered at the lines \( p_f(i) \) have only the points \( p_i \) and \( p_f(i) \) with \( P \) in common. The portion contained in the disk of any cone with center in the line \( p_f(i) \), axis \( p_i p_f(i) \), and with an opening angle of 2 arctan(\( \delta/2D \)); is a subset of the strip of width \( \delta \) around \( p_i p_f(i) \). Thus we can start with a small arbitrary value \( \epsilon_0 < \delta \) such that all points \( q_{j1}, q_{j2} \) are inside the disk. Obtain \( \epsilon_i \) as the distance from \( p_f(i) \) to the lines subtending the cone with center \( q_{i1} \), axis \( p_i p_f(i) \), and angle 2 arctan(\( \delta/2D \)). And finally set \( \epsilon \) as the minimum of all the \( \epsilon_i \).

From the halving-line matching definition we deduce that no two points in \( P \) are associated to the same halving line. This, together with the choice of \( \epsilon \), guarantees that \( Q \) is in general position whenever \( P \) is in general position. By construction the line \( q_{i1} q_{i2} \) is a halving line of \( Q \). In addition, since \( q_{i1} \) is in the interior of the triangle \( q_{i2}, q_f(i)1, q_f(i)2 \) the line \( q_{i1} q_f(i)1 \) (and also the line \( q_{i1} q_f(i)2 \)) is a halving line of \( Q \). Then
\[
\{(q_{i1}, \tilde{q}_{i1} q_f(i)1) : i = 1, 2, \ldots, N\} \cup \{(q_{i2}, \tilde{q}_{i1} q_f(i)2) : i = 1, 2, \ldots, N\}
\]
is a halving-line matching of \( Q \).
Now we proceed to calculate $\tau(Q)$ by counting crossings according to three different types. This method of counting was originally used by Aichholzer et al. [3], to calculate the number of crossings in their lens-replacement construction.

**Type I** Two points in pair $i$ and two in pair $j \neq i$. There are $\binom{N}{2}$ ways of choosing pairs $i$ and $j$ and all of them determine a crossing except when $j = f(i)$ or $i = f(j)$. Since there are exactly $N$ pairs $(i, f(i))$, the total number of crossings in this case is $\binom{N}{2} - N$.

**Type II** Two points in pair $i$ and the two other in pairs $j$ and $k$ all pairs distinct. First there are $N$ choices for the pair $i$. Then we have two cases, when $j, k \neq f(i)$, and when either $j$ or $k$ equals $f(i)$. In the first case (see Fig. 2, Type II(1)), to have a crossing, both $p_j$ and $p_k$ must be on the same side of the line $q_1 q_2$. Thus there are $2\left(\frac{N-1}{2}\right)$ ways of choosing $j$ and $k$ and then 4 choices for the second indices $x$ and $y$ for the points $q_{jx}$ and $q_{ky}$. In the second case (see Fig. 2, Type II(2)) we can assume $j = f(i)$. Then there are 2 choices for the second index $x$ of $q_{jx}$. Again, to have a crossing, we need $q_{jx}$ and $p_k$ on the same side of $q_1 q_2$. So there are $N/2 - 1$ ways of choosing $k$ and 2 choices for the second index $y$ of $q_{ky}$. The total number of Type II crossings is $N\left(\frac{N-1}{2}\right) + 4\left(\frac{N-1}{2}\right)$.

**Type III** Each point in a different pair. To have a crossing each of the four pairs must come from a crossing in $P$, so there are $\tau(P)$ possible pairs, and there are 2 choices for the second index in each pair. Thus there are 16 $\tau(P)$ number of crossings of Type III.

The list of crossings types is complete since by construction none of the segments $q_{i1} q_{i2}$ participate in any crossing. Adding Types I–III yields the result. $\square$

The strength of this lemma can be easily seen when it is applied to the set $P_6$ that minimizes $\tau(6) = 3$. We obtain $\tau(12) \leq \tau(Q) = 153$ which happens to be the correct value of $\tau(12)$ (a drawing is shown in Fig. 1). It took some effort [3] to find this point set in the past. In addition, if we use the sets $P_{30}, P_{36}, P_{48}$ obtained by Aichholzer et al., and reported in web page [2],
we obtain sets $P_{60}, P_{72}, P_{96}$ with less number of crossings than those previously known (see Table 1).

### 3. Proof of Theorems 1 and 2

Let $S_0 = P$ and $S_{k+1} = Q(S_k)$ for $k \geq 0$, where $Q(S_k)$ is the set given by Lemma 3. We first prove by induction that for $k \geq 0$,

$$
\overline{c}(S_k) = 16^k \overline{c}(P) + N^38^{k-1}(2^k - 1) - \frac{7}{6}N^24^{k-1}(4^k - 1) + \frac{5}{14}N2^{k-1}(8^k - 1).
$$

(2)

This identity is trivially true when $k = 0$. By Lemma 3 we have that $|S_k| = 2^k N$ and

$$
\overline{c}(S_{k+1}) = 16\overline{c}(S_k) + \frac{|S_k|}{2}(2|S_k|^2 - 7|S_k| + 5)
$$

$$
= 16\overline{c}(S_k) + N^38^k - \frac{7}{2}N^24^k + \frac{5}{2}N2^k.
$$

Then by induction hypothesis (2) we get

$$
\overline{c}(S_{k+1}) = 16^{k+1} \overline{c}(P) + N^38^k(2^{k+1} - 2) - \frac{7}{6}N^24^k(4^{k+1} - 4) + \frac{5}{14}N2^{k}(8^{k+1} - 8)
$$

$$
+ N^38^k - \frac{7}{2}N^24^k + \frac{5}{2}N2^k
$$

$$
= 16^{k+1} \overline{c}(P) + N^38^k(2^{k+1} - 1) - \frac{7}{6}N^24^k(4^{k+1} - 1) + \frac{5}{14}N2^{k}(8^{k+1} - 1),
$$

which proves (2) for all $k \geq 0$. Letting $n = |S_k| = 2^k N$ identity (2) becomes

$$
\overline{c}(n) \leq \overline{c}(S_k) = \left(\frac{24 \overline{c}(P) + 3N^3 - 7N^2 + (30/7)N}{24N^4}\right)n^4 - \frac{1}{8}n^3 + \frac{7}{24}n^2 - \frac{5}{28}n.
$$

(3)

This proves Theorem 2 for $n = 2^k N$. To establish the result for general $n$ we first show that $\overline{c}(n)/\binom{n}{4}$ is an increasing sequence. Indeed, if $|A| = n$ and $\overline{c}(n) = \overline{c}(A)$ then

$$(n-4)\overline{c}(n) = (n-4)\overline{c}(A) = \sum_{B \subseteq A} \overline{c}(B) \geq n \cdot \overline{c}(n-1)$$

and consequently $\overline{c}(n)/\binom{n}{4} \geq \overline{c}(n-1)/\binom{n-1}{4}$.

Suppose now that $2^k N \leq n < 2^{k+1} N$. From (3) we obtain $\overline{c}(2^{k+1} N) \leq c\binom{2^{k+1} N}{4} + c_1(2^{k+1} N)^3$ where $c$ is the coefficient of $\binom{n}{4}$ in Theorem 2 and $c_1$ is constant. Then

$$
\overline{c}(n) \leq \frac{\overline{c}(2^{k+1} N)\binom{n}{4}}{\binom{2^{k+1} N}{4}} \leq c\binom{n}{4} + 8c_1(2^k N)^3 \leq c\binom{n}{4} + 8c_1 n^3
$$

Table 1

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\overline{c}(P_n)$</th>
<th>$\overline{c}(Q(P_n))$</th>
<th>Old bound for $\overline{c}(2n)$</th>
<th>New bound for $\overline{c}(2n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>9726</td>
<td>179,541</td>
<td>$\leq 179,544$</td>
<td>$\leq 179,541$</td>
</tr>
<tr>
<td>36</td>
<td>21,175</td>
<td>381,010</td>
<td>$\leq 381,020$</td>
<td>$\leq 381,010$</td>
</tr>
<tr>
<td>48</td>
<td>71,028</td>
<td>1,239,096</td>
<td>$\leq 1,239,139$</td>
<td>$\leq 1,239,096$</td>
</tr>
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</table>
Table 2
Coordinates of point set $P_{30}$ with 9776 crossings

<table>
<thead>
<tr>
<th>$i$th point = $p_i$ = (x-coordinate, y-coordinate)</th>
<th>$p_1$</th>
<th>$p_2$</th>
<th>$p_3$</th>
<th>$p_4$</th>
<th>$p_5$</th>
<th>$p_6$</th>
<th>$p_7$</th>
<th>$p_8$</th>
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<tbody>
<tr>
<td>$p_1$ = (9259, 16, 598)</td>
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<td>$p_2$ = (9763, 16, 199)</td>
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<td>$p_3$ = (9977, 16, 397)</td>
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<td>$p_4$ = (10, 248, 16, 225)</td>
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<td>$p_5$ = (10, 666, 16, 385)</td>
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<td>$p_6$ = (12, 849, 16, 335)</td>
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<td>$p_7$ = (18, 577, 16, 451)</td>
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<td>$p_8$ = (10, 391, 16, 281)</td>
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<tr>
<th>$i$th point = $p_{17}$ = (x-coordinate, y-coordinate)</th>
<th>$p_9$</th>
<th>$p_{10}$</th>
<th>$p_{11}$</th>
<th>$p_{12}$</th>
<th>$p_{13}$</th>
<th>$p_{14}$</th>
<th>$p_{15}$</th>
<th>$p_{16}$</th>
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<tr>
<td>$p_9$ = (28, 477, 16, 613)</td>
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<td>$p_{10}$ = (15, 909, 16, 415)</td>
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<td>$p_{11}$ = (9446, 15, 905)</td>
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<td>$p_{12}$ = (9540, 16, 541)</td>
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<td>$p_{13}$ = (9262, 16, 627)</td>
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<td>$p_{14}$ = (9282, 16, 947)</td>
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<td>$p_{15}$ = (8912, 17, 261)</td>
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<td>$p_{16}$ = (7842, 19, 232)</td>
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<tr>
<th>$i$th point = $p_{25}$ = (x-coordinate, y-coordinate)</th>
<th>$p_{17}$</th>
<th>$p_{18}$</th>
<th>$p_{19}$</th>
<th>$p_{20}$</th>
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<tr>
<td>$p_{17}$ = (5141, 23, 755)</td>
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<td>$p_{18}$ = (9154, 17, 055)</td>
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<td>$p_{19}$ = (0, 32, 394)</td>
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<td>$p_{20}$ = (6820, 20, 921)</td>
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<td>$p_{21}$ = (9949, 16, 415)</td>
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<td>$p_{23}$ = (9419, 15, 893)</td>
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<td>$p_{24}$ = (9146, 15, 771)</td>
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<td>$p_{25}$ = (9075, 15, 320)</td>
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Table 3
Halving-line matching of $P_{30}$

<table>
<thead>
<tr>
<th>$\alpha$, $(a, b)$ means $(p_a, p_b p_b)$</th>
<th>$1$, $(1, 3)$</th>
<th>$9$, $(9, 13)$</th>
<th>$17$, $(17, 18)$</th>
<th>$25$, $(25, 26)$</th>
</tr>
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<tbody>
<tr>
<td>$2$, $(2, 4)$</td>
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<tr>
<td>$3$, $(3, 7)$</td>
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<tr>
<td>$4$, $(4, 5)$</td>
<td></td>
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<tr>
<td>$5$, $(5, 8)$</td>
<td></td>
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<tr>
<td>$6$, $(6, 7)$</td>
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<tr>
<td>$7$, $(7, 9)$</td>
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<tr>
<td>$8$, $(8, 10)$</td>
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</tbody>
</table>

which proves Theorem 2. Now, to prove Theorem 1, consider the set $P = P_{30}$ with coordinates in Table 2 obtained from [2]. It satisfies that $c(P_{30}) = 9726$ and it can be verified that the set of point-line pairs in Table 3 represents a halving-line matching of $P_{30}$. We then get

$$c(P) \leq \frac{29,969}{78,750} \binom{n}{4} + \Theta(n^3).$$

References

