Antimagic Labellings of Graphs
DIMACS REU Student Presentation 1

Timothy Reynolds

Mentor: Tao-Ming Wang

June 11, 2010
Graph Labellings

Definition

A labelling of a graph is an assignment of numbers, set elements, or group elements to the vertices or edges of the graph that satisfies some properties.

In this presentation, we will take graphs to be finite, simple, and connected.
Definition

A labelling of a graph is an assignment of numbers, set elements, or group elements to the vertices or edges of the graph that satisfies some properties.
Definition
A labelling of a graph is an assignment of numbers, set elements, or group elements to the vertices or edges of the graph that satisfies some properties.

In this presentation, we will take graphs to be finite, simple, and connected.
One type of labelling is called a magic labelling. In a magic labelling, the edges of the graph are labelled distinctly from 1 to $|E|$ such that the sum of the edges at each vertex is the same.
One type of labelling is called a magic labelling.
One type of labelling is called a **magic labelling**.

<table>
<thead>
<tr>
<th></th>
<th>8</th>
<th>1</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>5</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>9</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>
One type of labelling is called a magic labelling.

\[
\begin{array}{ccc}
8 & 1 & 6 \\
3 & 5 & 7 \\
4 & 9 & 2 \\
\end{array}
\]

In a magic labelling, the edges of the graph are labelled distinctly from 1 to \( |E| \) such that the sum of the edges at each vertex is the same.
Example of Magic Labelling

Figure: $K_{3,3}$: The complete bipartite graph on 3 and 3 vertices
Example of Magic Labelling

Figure: A magic labelling of $K_{3,3}$

Each vertex sum is equal to 15.
Example of Magic Labelling

Figure: A magic labelling of $K_{3,3}$

Each vertex sum is equal to 15.
This is related to the magic square: the top vertices correspond to the rows and the bottom vertices to the columns.
Antimagic Labelling

In contrast, an antimagic labelling is one in which the vertex sums are all different. More precisely, for a graph \((V, E)\), an antimagic labelling is a bijection \(f : E \to \{1, 2, \ldots, |E|\}\) such that, for any two distinct vertices \(v_1, v_2 \in V\), we have

\[\sum_{e_i \in E \text{ incident to } v_1} f(e_i) \neq \sum_{e_j \in E \text{ incident to } v_2} f(e_j)\]

We call a graph antimagic if it admits an antimagic labelling.
Antimagic Labelling

In contrast, an antimagic labelling is one in which the vertex sums are all different.
Antimagic Labelling

In contrast, an antimagic labelling is one in which the vertex sums are all different.
More precisely, for a graph \((V, E)\), an antimagic labelling is a bijection

\[
f : E \rightarrow \{1, 2, \ldots, |E|\}
\]

such that, for any two distinct vertices \(v_1, v_2 \in V\), we have

\[
\sum_{e_i \in E \text{ incident to } v_1} f(e_i) \neq \sum_{e_j \in E \text{ incident to } v_2} f(e_j)
\]
Antimagic Labelling

In contrast, an antimagic labelling is one in which the vertex sums are all different. More precisely, for a graph \((V, E)\), an antimagic labelling is a bijection

\[ f : E \rightarrow \{1, 2, \ldots, |E|\} \]

such that, for any two distinct vertices \(v_1, v_2 \in V\), we have

\[
\sum_{e_i \in E \text{ incident to } v_1} f(e_i) \neq \sum_{e_j \in E \text{ incident to } v_2} f(e_j)
\]

We call a graph antimagic if it admits an antimagic labelling.
Example Antimagic Labelling

Figure: $K_4$: The complete graph on 4 vertices
Example Antimagic Labelling

Figure: A labelling of $K_4$...
Example Antimagic Labelling

Figure: A labelling of $K_4$ which is antimagic
What graphs are antimagic?
What graphs are antimagic?

- Graphs with at least 4 vertices and a vertex of degree $|V| - 1$ or $|V| - 2$ (Alon et al 2004)
What graphs are antimagic?

- Graphs with at least 4 vertices and a vertex of degree $|V| - 1$ or $|V| - 2$ (Alon et al 2004)
- Graphs with minimum degree $C \log |V|$ for some constant $C$ (Alon et al 2004)
What graphs are antimagic?

- Graphs with at least 4 vertices and a vertex of degree $|V| - 1$ or $|V| - 2$ (Alon et al 2004)
- Graphs with minimum degree $C \log |V|$ for some constant $C$ (Alon et al 2004)
- Regular bipartite graphs except $K_2$ (Cranston 2009)
What graphs are antimagic?

- Graphs with at least 4 vertices and a vertex of degree $|V| - 1$ or $|V| - 2$ (Alon et al 2004)
- Graphs with minimum degree $C \log |V|$ for some constant $C$ (Alon et al 2004)
- Regular bipartite graphs except $K_2$ (Cranston 2009)
- Complete partite graphs except $K_2$ (Alon et al 2004)
What graphs are antimagic?

- Graphs with at least 4 vertices and a vertex of degree $|V| - 1$ or $|V| - 2$ (Alon et al 2004)
- Graphs with minimum degree $C \log |V|$ for some constant $C$ (Alon et al 2004)
- Regular bipartite graphs except $K_2$ (Cranston 2009)
- Complete partite graphs except $K_2$ (Alon et al 2004)
- $C_m \times C_n$ (Wang 2005), $C_m \times P_n$, $P_m \times P_n$ (Cheng 2006)
What graphs aren’t antimagic?
What graphs aren’t antimagic?

- $K_2$
What graphs aren’t antimagic?

- $K_2$

**Figure:** The only labelling of $K_2$
What graphs aren’t antimagic?

- $K_2$

Figure: The only labelling of $K_2$

- Any others?
What graphs are antimagic?

Conjecture

All finite, simple, connected graphs except $K_2$ are antimagic.
(Hartsfield and Ringel, 1990)
What graphs are antimagic?

Conjecture

All finite, simple, connected graphs except $K_2$ are antimagic. (Hartsfield and Ringel, 1990)

I am working towards proving this conjecture for regular graphs with perfect matchings.
Other types of labellings

Definition
An \((a,d)\)-antimagic labelling is an antimagic labelling in which the vertex sums form an arithmetic progression.
Other types of labellings

Definition
An \((a, d)\)-antimagic labelling is an antimagic labelling in which the vertex sums form an arithmetic progression.

It is conjectured that the generalized Petersen graphs \(P(n, k)\) are \((a, d)\)-antimagic for some \(a\) and \(d\) (Miller and Bača 2000).
Other types of labellings

Definition
A $\mathbb{Z}_k$-magic labelling is an assignment of the elements of the group $\mathbb{Z}_k \setminus \{0\}$ to the edges of a graph such that the sum at each vertex (in $\mathbb{Z}_k$) is constant.
Other types of labellings

Definition

A $\mathbb{Z}_k$-magic labelling is an assignment of the elements of the group $\mathbb{Z}_k \setminus \{0\}$ to the edges of a graph such that the sum at each vertex (in $\mathbb{Z}_k$) is constant.

For a given graph, which constants can be achieved?
Other types of labellings

Definition

A $\mathbb{Z}_k$-magic labelling is an assignment of the elements of the group $\mathbb{Z}_k \setminus \{0\}$ to the edges of a graph such that the sum at each vertex (in $\mathbb{Z}_k$) is constant.

For a given graph, which constants can be achieved?
For what graphs can all constants be achieved?