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The Triangle Algorithm:
A Geometric Approach to Systems of Linear Equations
Definition:
Given a finite set of points $S$ in $\mathbb{R}^m$, the convex hull is the minimal convex set containing $S$.

The convex hull of a finite point set $S$ is the set of all convex combinations of its points. Written as:

$$conv(S) = \left\{ \sum_{i=1}^{\left|S\right|} \alpha_i x_i \mid (\forall i : \alpha_i \geq 0) \land \sum_{i=1}^{\left|S\right|} \alpha_i = 1 \right\}.$$

The convex hull of a finite point set in $\mathbb{R}^m$ forms a convex polygon.
The Convex Hull Decision Problem

Given a finite set of points in $\mathbb{R}^m$, we must first determine whether or not a distinguished point, $p$, is in the convex hull.

How do we accomplish this? First, an important theorem:

Distance Duality Theorem: Let $S = \{v_1, ..., v_n\} \subset \mathbb{R}^m$, $p \in \mathbb{R}^m$. Let $d(u, w)$ denote the Euclidean distance between the points $u$ and $w$.

i) $p \in conv(S)$ if and only if given any $p' \in conv(S)$, there exists $v_j$ such that $d(p', v_j) \geq d(p, v_j)$.

ii) $p$ is not in the $conv(S)$ if and only if there exists $p' \in conv(S)$ such that $d(p', v_j) < d(p, v_j), \ \forall \ i$. In this case we call $p'$ a witness. (B. Kalantari 2012)
The Triangle Algorithm

- Given a $p'$, search for a triangle $\triangle pp'v_j$ where $v_j \in S$, $p' \in \text{conv}(S) \setminus \{p\}$, such that $d(p', v_j) \geq d(p, v_j)$. The point $p'$ is the **iterate**
- If $\triangle pp'v_j$ satisfies the inequality above, $v_j$ becomes a **pivot point** to “pull” the current iterate, along the line segment, closer to $p$ to get a new iterate $p'' \in \text{conv}(S)$.
  
  Note: Since $p''$ is a convex combination of $p'$ and $v_j$, it will remain in $\text{conv}(S)$.
- If no such triangle exists, then $p'$ is a **witness** certifying that $p$ is not in $\text{conv}(S)$. 
Application to systems of linear equations:

- Think of $Ax = b$ as $A\alpha = p$, where the columns of $A$ are the vertices, \( \alpha \) is a column of convex coefficients, and \( p \) is our distinguished point.
  - Suppose $A$ is $nxn$ invertible, and $x = A^{-1}b \geq 0$. It follows that solving this system is equivalent to approximating $0 \in \text{conv}(a_1, a_2, \ldots, a_n, -b)$.
  - What if we don’t know if $x$ is nonnegative? Let $e = (1, \ldots, 1)^T \in \mathbb{R}^m$. Then there exists a $t \geq 0$ such that if $x$ is a solution to $A(x - te) = b$, then $x = A^{-1}b + A^{-1}(Ate) \geq 0$. We may apply the convex hull approach.

Application to eigenvalue problems will be considered
Methodology

- Test the Triangle Algorithm against traditional methods of solving systems of linear equations:
  - Examples: Jacobi iteration, Gauss-Seidel Method
    - Possibly Krylov Subspace Methods:
  - Implementation through Matlab
  - We will be looking at improving convergence rates and comparing computational efficiency.
Implementing Dr. Kalantari’s Triangle Algorithm presents a unique and exciting new way at looking at solving systems of linear equations.

- Applications in linear programming
- Possible applications in Eigenvalue problems may also be considered
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References

- B. Kalantari. A Characterization Theorem and an Algorithm for a Convex Hull Problem.