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Co-evolution of Opinion and Signed Network Dynamics in Real-World Scenarios

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ABSTRACT
With the pervasion of social media in modern society, evolution of opinions and network dynamics in social networks have gained special interest from industry, academia and governments. The study of opinion and network dynamics has several applications- social sciences, behavioral economics, game theory, decentralized robot swarms in smart cities to name a few- with consequential implications. Adversarial attacks through fake news and troll bots are increasingly common and there is a need for a rigorous understanding of their impact on individuals and the social networks. In this paper, we study opinion and network dynamics and characterize social networks at their limit states extending recent work in co-evolution of signed networks and opinions to better represent the real world. We build upon intuition of the real world, in which individuals hold multiple opinions on a range of issues for example- healthcare, gun control and seek to better represent complex relationships/social ties between individuals in a network. We formulate these intuitions mathematically and incorporate them in a co-evolution model which combines signed network dynamics based on structural theory and opinion dynamics. We conduct a comprehensive study through numerous simulations and validate the model on the benchmark Zachary's Karate Club dataset to achieve 100% accuracy. We mathematically address divergence of social ties to infinity, which in the real world would translate to limitless polarization, by introducing normalization which keeps the relative magnitude of the social ties in tact but prevents divergence.

INDEX TERMS Computational Social Science, Information Warfare, Networks, Network Dynamics, Opinion Dynamics, Robot Swarms, Social Networks, Social Science

I. INTRODUCTION

MODERN technology has revolutionized the way we consume information and interact with people around the world. Today, social media platforms allow for near instantaneous exchange of information and opinions to vast audiences through the click of a button- a luxury enjoyed previously only by traditional media outlets. As a result, the information that we are exposed to has risen by orders of magnitude. Yet, just 65.6% of the world has access to internet, and with this number expected to grow by the day, the interaction between individuals and consumption of information is only expected to increase. While the information that we are subject to has vastly increased, our cognitive processing capabilities have remained largely unchanged leading to an information overload and the consequent distortion of facts and ultimately to the erosion of absolute truth. It thus becomes necessary to better understand the social processes that underlie the complex interactions and consequent exchange of opinions/information and its side effects.

Recent events in the political domain further necessitate the need to gain a comprehensive understanding of the social dynamics underlying the social networks which may be exploited by malign actors (both internal and external) thereby posing a threat to national security. For example, the rise of troll armies/bots on almost every social media platform and the planting of fake news requires a comprehensive evaluation of their impact on manipulating crowd behavior. Twitter disclosed in 2019 that external bots had made a coordinated effort to sow political discord in Hong Kong. U.S. Special Counsel's office argues Russian IRA (Internet Research Agency) made a strategic effort to sow discord in the U.S. political system. Recent work by Gaitonde et al. [1] investigates the ability of an adversary to artificially induce disagreement/discord and qualitatively describes structural properties of networks which are susceptible to and resilient against adversarial attacks. They also consider the uncoupling of social dynamics processes where the opinion dynamics take place on a network (eg. social media platforms) that is different from the network on which disagreement is measured (eg. 'real-world connections').

Understanding the processes underlying crowd behavior and their manipulation also has applications in engineering problems of modern smart cities. Robot swarms are expected to play a central role in the smart cities of the future with innumerable applications in optimizing- garbage collection/disposal, precision farming, search operations etc. as shown in A. L. Alfeo et al. [2]. With larger and larger robot swarms, it becomes computationally efficient to have decentralized control i.e. the robots "communicate" with each other independently without global coordination (analogous to human interaction) but exhibit globally desirable behavior (eg. avoid splitting of the swarm). This is analogous to understanding the social processes that lead to polarization/harmony in humans.

With the vast amount of information that individuals are subject to, opinions of individuals are influenced to a large extent by their social networks and the strength of the ties in the network, in turn, are influenced by the interactions of individuals holding various opinions.

Most work in this area has focused on models of opinion dynamics or on models of network dynamics but not on models of both opinion and network dynamics. In this paper we extend a co-evolution model (of both opinion and network dynamics) as presented in Wang et al. [3] to better reflect real world settings and validate it on real world data...

II. BACKGROUND AND RELATED WORKS

Opinion dynamics is the interplay of between opinion formation and the network structure of interactions. Dynamics of opinions and social ties (network dynamics) have been extensively studied and several mathematical models proposed that aim to capture wholly or partially the complex social processes taking place.

Formation of community structures is a widely observed phenomena in social networks and have been investigated in theory and real world platforms [4,5,6]. In this paper, we extend the the model in [3] and validate findings that show how network and opinion evolution leads to emergence of community structure.

To provide context regarding previous models, we define a network of individuals represented as an adjacency matrix W with each entry $w_{ij} \in \mathbb{R}$ representing the strength of the social tie/influence between individual i and j. We also consider an opinion vector $\mathbf{v}$ with each entry $\mathbf{v}_i$ representing the opinion of individual i.

One of the first models of opinion dynamics is the French Degroot model which considers a discrete time process of opinion for a group of n individuals. The elements of the adjacency matrix or social ties, w_{ij} are non-negative. The opinions are updated according to

$$V(t+1) = WV(t), \text{ where } W \text{ is a stochastic matrix}$$

Abelson's model [10] defines the analogous continuous-time model.

The Friedkin-Johnson model, in addition to having W and V as defined above, where W is a stochastic matrix, considers a diagonal influence matrix Λ such that $\lambda_i \in [0, 1]$ and λ_i defines the susceptibility of an individual to social influence.

In most cases, these models consider a constant weight matrix W and even those models in which W is time varying such as Hegselmann-Krause [11, 12], Deffuant [13] and W co-evolves with the opinions W contains only non-negative entries representing positive and neutral/no influence between individuals. So, any interaction between individuals in these networks moves their opinions closer to each other.

Networks with positive and negative ties- signed networks have been extensively studied as well. A fundamental characterization of a signed network is whether or not it exhibits structural balance.

Some recent work presents a co-evolving model, Altafini [14] in which the weight matrix W is fixed but contains positive and negative values. Still, the weight matrix is independent of the opinion changes. In A Proskornikov, M Cao et al [15, 16], the network/social ties vary with time, but the evolution of the network is still independent of opinion changes.

Wang et al. in [3] present a co-evolution model in which w_{ij} does not need to be non-negative and takes values in $(-\infty, +\infty)$. The model is presented in the next section and combines signed network dynamics using structural balance theory [8], described in the next section, and classical opinion dynamics in which the network dynamics are influence by the opinion dynamics.

We extend the co-evolution model which combines the dynamics of signed networks shown in [3] to better reflect intuitions of individuals holding multiple opinions and address limitless polarization/harmony seen in the simulations using normalization to better reflect the real world.

III. DESIGN AND MODEL

We first discuss some required preliminaries to discuss the model. We then introduce the discrete-time co-evolution model as presented in Wang et al. [3] and then discuss our additions to the model.

A. PRELIMINARIES

We define a network of individuals represented as an adjacency matrix W with each entry $w_{ij} \in \mathbb{R}$ representing the strength of the social tie/influence between individual i and j. A positive value would indicate that i and j and friends while a negative value would indicate that they are hostile. A value of 0 for w_{ij} means that individuals i and j are not related. We also consider an opinion vector $\mathbf{v}$ with each entry $\mathbf{v}_i \in [-1, 1]$ representing the opinion of individual i. Each vertex i in the network has an opinion $\mathbf{v}_i$ and weight w_{ij} between individual i and j both of which are updated at each time step representing the evolution of the system. The goal is to characterize the states of the dynamic system at the limit i.e. whether there is emergence of community structure- which is indicated by partitioning of the network into two groups with positive edges inside each group and negative edges across- and structural balance.

The extreme case of community structure is indicated by structural balance [8] in networks where there are two types of social ties such as friendship and hostility represented by positive and negative values respectively. In social networks, only two types of triangles are stable- triangles in which all the ties are positive (indicating everyone is friends) and triangle in which there are two negative edges and one positive edge depicting that the enemy of your enemy is your friend (Fig 1). Over time, unstable triangles break and change into stable ones throughout the network. The stable network theory states that a network in which all the triangles are stable must have a partition of nodes into two camps- thus having community structure.

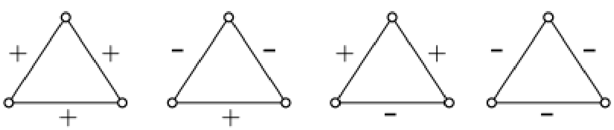


FIGURE 1. The two triangles on the right indicate structural balance

B. CO-EVOLUTION MODEL

Suppose that there are n individuals in the network each having their own opinion. This may represent their opinion on a certain issue such as for/against gun control corresponding to 1 and -1 respectively. The opinion of all the individuals is the vector $\mathbf{v} \in \mathbb{R}^n$. We also have an initial influence matrix W in which $w_{ij} \in \mathbb{R}$ represents the relationship between individuals i and j. W is assumed to be a symmetric matrix indicating an undirected network. The dynamic system dictating the evolution of the social ties/weight matrix W and the opinions/opinion vector $\mathbf{v}$ in the social network as presented in [3] is:

$$V(t+1) = V(t) + W(t)V(t) \tag{1}$$

$$W(t+1) = W(t) + V(t)V(t)^T \tag{2}$$

In equation (1), the opinion of the individual changes by the weighted sum of the opinions of its neighbors with coefficients from the weight matrix W. In equation (2), the weights or the strength of the social ties between individuals i and j changes by the difference in opinions of individuals i and j.

C. DESIGN

In the co-evolution model in [3], the authors define the co-evolution model such that each individual i in the network has a single opinion $v_i \in \mathbb{R}$. However, in reality individuals hold opinions for a wide variety of topics while agreeing and disagreeing over different topics. For example an individual may support 'gun control' and support 'free healthcare for all' whereas another individual may support 'gun control' but be against 'free healthcare for all'. Modelling multidimensional opinions and their complex interactions is more representative of the real world. To address this, we propose two ways to capture the multiple opinions that an individual may hold within the context of the model:

1) The weight matrix represents the influence between two individuals with a single real value i.e. W is a square matrix with each entry a real number while the opinion vector of the entire network $\mathbf{v}$ is such that each entry v_i in $\mathbf{v}$ is an element of $\mathbb{R}^{1x2}$ representing the two opinion and each individual holds on different topics.
2) The weight matrix represents the influence between two individuals with an m x m matrix (assuming each individual has m opinions). Here, each entry of the square matrix W is an m x m matrix. This allows us to model the interactions between opinions and the influence exerted by individuals on specific opinions as opposed to the just having a single real value for the influence between to individuals.

In case 2 as described above, each entry w_{ij} of the weight matrix representing the social tie between two individuals is a mxm matrix which can capture the interactions between different opinions of two individuals. As described above two individuals who agree on certain issues while differing on others are likely to have a complex relationship and their social relationship may not be adequately capture by a single real number. Furthermore, this disagreement/agreement of opinions on certain issues is likely to impact the opinion evolution in more complex ways as well.

To see this, consider the example of two individuals A and B. Suppose A strongly supports gun control i.e. $v_{a_1} = 1$ and B is unsure but is somewhat against gun control i.e. $v_{b_1} = -0.15$. Now suppose both A and B are strongly against free healthcare, i.e. $v_{a_2} = -1$ and $v_{b_2} = -1$. Suppose A and B are relatively friendly. Then using the co-evolution model, the weight/social tie between A and B could get reinforced considering there is strong agreement between A and B on free healthcare which in turn could sway A's weak opinion on gun control towards that of B's. Or perhaps, if A and B have a more hostile relationship on the topic of gun control, it could cause their entire relationship to turn hostile as the system evolves. Perhaps they may find middle ground while remaining hostile regarding issues of disagreement and friendly regarding issue of agreement. By using a matrix to capture the relationship between individuals in a network as opposed to real numbers, we hope to capture these complex interactions between opinions of individuals on different topics which is more representative of the real world.

The simulations of the co-evolution model presented in [3] (Fig 2) show that in all cases the values of the weight matrix W and the magnitude of the elements of the opinion vector $\mathbf{v}$ diverge to positive and negative infinity. What does this translate to in the real world? Does it mean that that societies/networks head towards complete partitioning into two camps which are then completely isolated? To better understand the behavior of the co-evolution model at limit points, we evaluate the co-evolution model under various normalization constraints i.e. the weight matrix W and the opinion vector $\mathbf{v}$ are normalized at each timestep. Normalization at each timestep leaves the relative values of the weights w_{ij} in W and opinions v_i in $\mathbf{v}$ unchanged while being bounded so as not to diverge to positive/negative infinity. We use various normalization techniques- Frobenius/Euclidean norm, infinity norm, 1-norm, 2-norm, dividing by the absolute maximum of the matrix/vector. The results for all the normalization techniques were consistent and there was no discernible difference.

IV. RESULTS AND DISCUSSION

We perform simulations for networks exhibiting various properties. For the multidimensional opinions setting i.e. where each individual holds more than one opinion corresponding to different topics, we limit the number of opinions an individual can hold to 2 for visualization purposes. W is weight matrix representing the social tie between two individuals and $\mathbf{v}$ is the vector representing the opinions of the individuals in the network.

A. CASE 1

These are the experiments for Case 1 as described in the Design subsection of the previous section. Here,

$$W = \begin{bmatrix} w_{11} & w_{12} & \dots & w_{1n} \\ w_{21} & \ddots & & \\ \vdots & & \ddots & \\ w_{n1} & \dots & \dots & w_{nn} \end{bmatrix}, \text{ where } w_{ij} \in \mathbb{R}$$

$$\mathbf{V} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \text{ where } v_i \in \mathbb{R}^{1x2} \text{ and } v_{i_1}, v_{i_2} \in \text{[-1,1]}$$

W is initialized to a random symmetric matrix representing an undirected graph and $\mathbf{v}$ is initialized randomly following the dimensions and constraints described above.

The results are shown in Figure 3. Here, the elements of the weight matrix or the social ties in the network diverge to positive and negative infinity through time. Similarly, the opinion vectors for each individual also diverge. This network contains 10 individuals for the sake of clarity in the visualization but similar results are observed for networks with higher individuals.

We then run the same network of 10 individuals under normalization constraints. The weight matrix and opinion vector are normalized by dividing by the absolution maximum of the weight matrix and opinion vector respectively. This prevents the values from going to infinity but does not change their relative magnitudes. The results are shown in Figure 4.

We see that the values of the weight matrix, which otherwise diverge to positive/negative infinity converge due to normalization. The opinion vector for each individual however oscillates as time goes on. The red vectors indicate the vector point was in the first quadrant and the blue vectors indicate the vector point was in the third quadrant of the euclidean space at the time the simulation was stopped. The oscillation of the opinion vectors is an interesting observation as this would correspond to a real world system in which community structure does not emerge and the opinions of individuals continue to vary through all possible range of values. We also run a simulation in which the weight matrix is normalized as described above at each iteration but the opinion vector is not. The result (Figure 4) shows that the elements of the weight matrix converge but the opinion vectors for each individual diverge indicating that the social ties remain constant while there is community formation with regards to opinions held by the individuals.

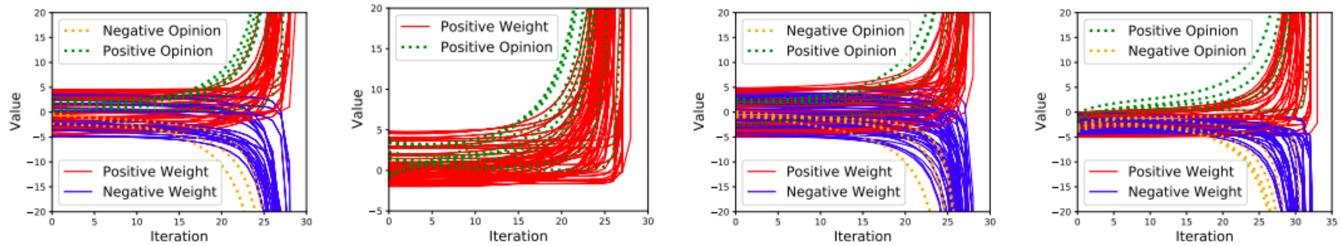
B. CASE 2

These are the experiments for Case 2 as described in the Design subsection of the previous section. Here,

$$W = \begin{bmatrix} w_{11} & w_{12} & \dots & w_{1n} \\ w_{21} & \ddots & & \\ \vdots & & \ddots & \\ w_{n1} & \dots & \dots & w_{nn} \end{bmatrix}, \text{ where } w_{ij} \in \mathbb{R}^{2x2}$$

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \text{ where } v_i \in \mathbb{R}^{2x1} \text{ and } v_{i_1}, v_{i_2} \in \text{[-1,1]}$$

W is initialized to a random symmetric matrix representing an undirected graph and $\mathbf{v}$ is initialized randomly following the dimensions and constraints described above.



(a) Symmetric matrix $W(0)$ (b) Symmetric matrix $W(0)$ (c) Random matrix $W(0)$ (d) Negative matrix $W(0)$

FIGURE 2. Results from [3]. The magnitudes of the elements of the weight matrix and opinion vector go to positive/negative infinity

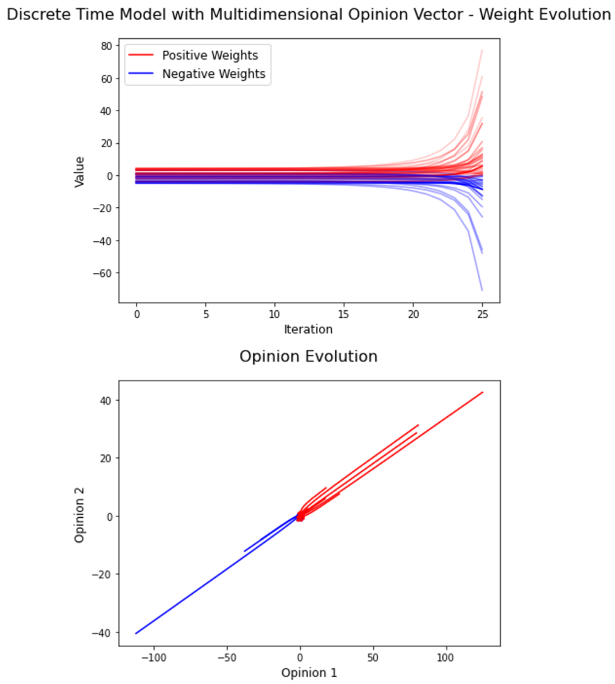


FIGURE 3. The magnitudes of the elements of the weight matrix (top) diverge to positive and negative infinity. The opinion vector of each individual diverge (bottom)

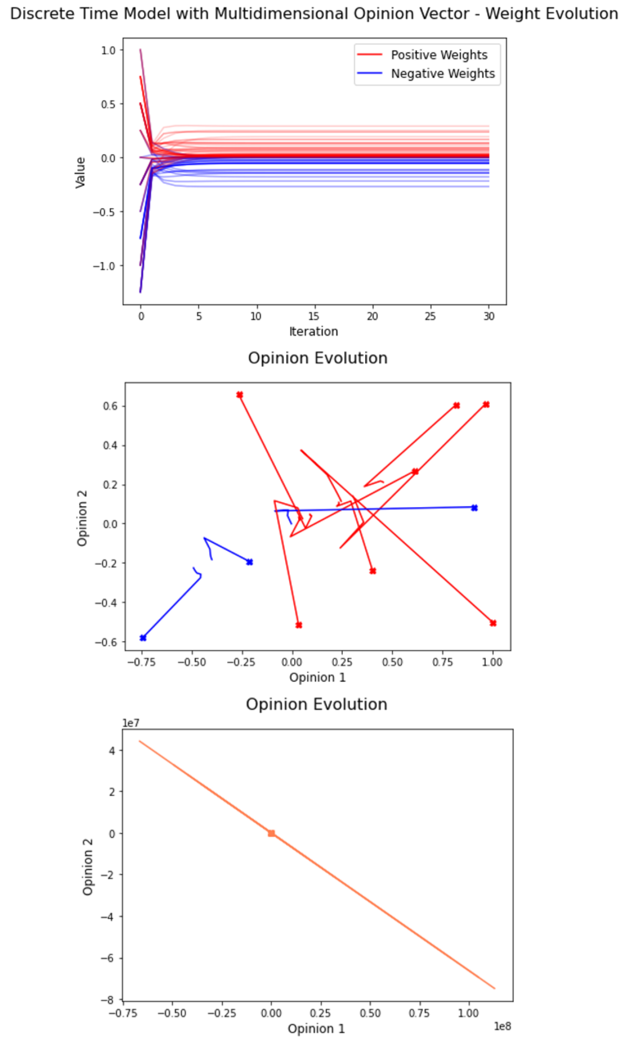


FIGURE 4. The magnitudes of the elements of the weight matrix (top) converge due to normalization. The opinion vector of each individual oscillates as time goes on (middle). The initial opinion vector for each individual is denoted by the '+' sign. (Bottom) shows the divergence of the opinion vectors when the weight matrix is normalized at each iteration but the opinion vector is not.

The results are shown in Figure 5. Here, the elements of the weight matrix or the social ties in the network diverge to positive and negative infinity through time but are not shown in the visualization since it the ties are represented by a matrix. Similarly, the opinion vectors for each individual also diverge. This time however, the opinion vectors go to infinity in multiple directions as opposed to 2 in Case 1. This model thus captures more complex interactions between individuals allowing them to hold various combinations and degrees of opinions on the two issues. This network contains 7 individuals for the sake of clarity in the visualization but similar results are observed for networks with higher individuals.

We then run the same network of 7 individuals under normalization constraints. The weight matrix and opinion vector

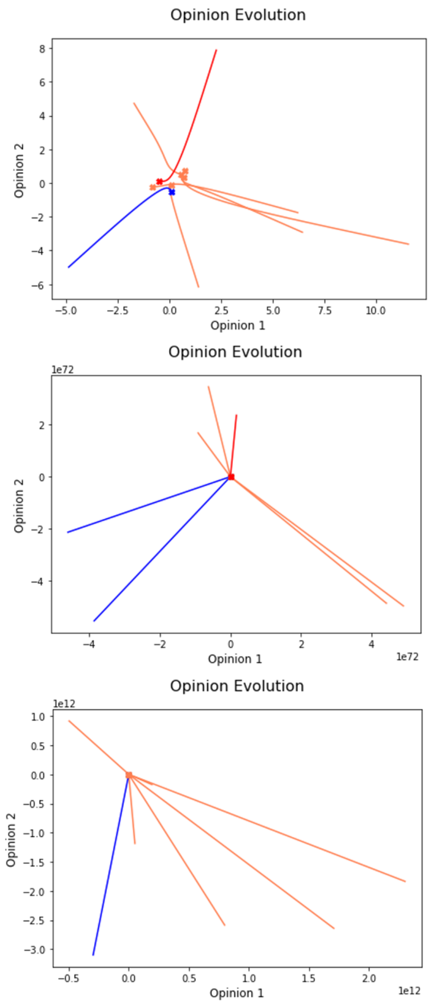


FIGURE 5. The figure shows 3 different simulations (Top, Middle, Bottom)

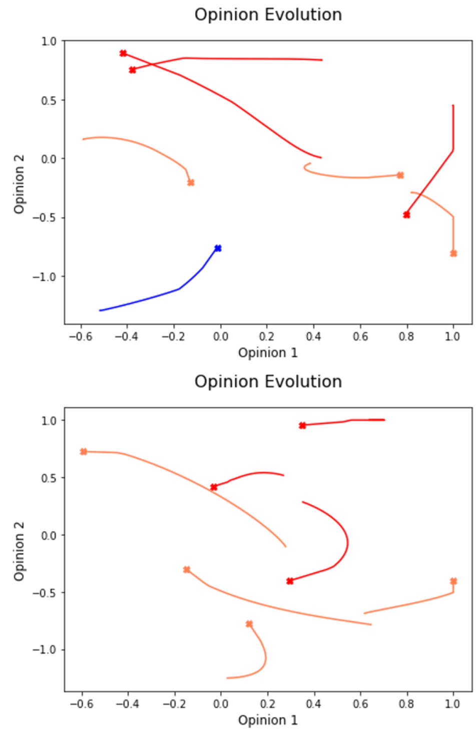


FIGURE 6. The figure shows 2 different simulations (Top, Bottom) for Case 2 with normalization. The dots on the lines denote the starting point of the vectors on in Euclidean space

are normalized by dividing by the absolution maximum of the weight matrix and opinion vector respectively. This prevents the values from going to infinity but does not change their relative magnitudes. The results are shown in Figure 6.

Similar to the results in case 1, we see that the opinion vectors of the individuals oscillate as time goes on.

C. REAL WORLD DATA

We validate our extension of the co-evolution model using the benchmark Zachary's Karate Club dataset [9]. In this study, the author witnesses the breakup of 34 members of a karate club over the span of two years into two factions. the author closely documents the links of external interactions between members of the club i.e. interactions outside the club.

Conflict arose between two members of the club- node 0 and node 33 as shown in Figure 8 leading to a split among the club members.

To model this, we set

$$\mathbf{v} = \begin{bmatrix} [1 & 1] \\ [0 & 0] \\ \vdots & \\ [0 & 0] \\ [-1 & -1] \end{bmatrix}$$

Since node 0 and node 33 are the nodes between which the conflict first arose, we set them to have opinion vectors $v_0 = [1\ 1]$ and $v_1 = [-1\ \ -1]$ to indicate complete disagreement on two issues. Every other nodes i is initialized to be indifferent and defined $v_i = [0\ \ 0]$.

The weight matrix W, is set according to the adjacency matrix of the graph where $w_{ij} = 1$ if two individuals interacted outside of the club and 0 otherwise.

The systems evolves according to the co-evolution model. We normalize the weight matrix at each iteration of the co-evolution model. The results are shown in Figure 7.

We see that community structure emerges- the nodes split into two camps as indicated by the opinion vectors diverge in opposite directions. Since the weight matrix is normalized, we see that the social ties converge but the opinions remain polarized as time goes on. The resulting community formation correctly classifies each node in agreement with the ground truth i.e. the actual split of the members as documented by Zachary. The result show that the co-evolution model with multidimensional opinion vectors and normalization is to some degree representative of the real

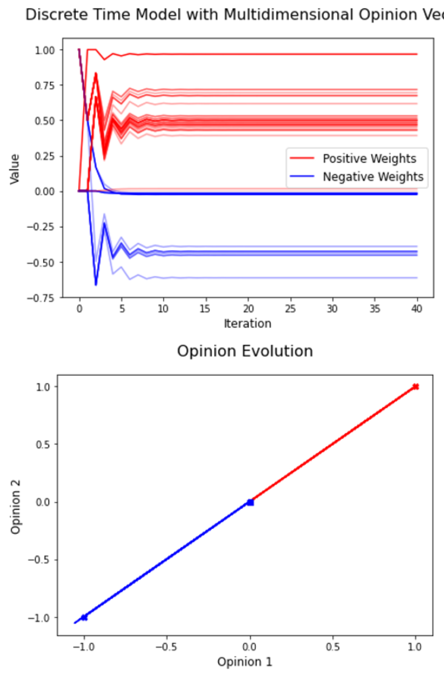


FIGURE 7. Graph of interactions between members of the Karate club as documented by Zachary. Red and Green nodes indicate the split into two factions

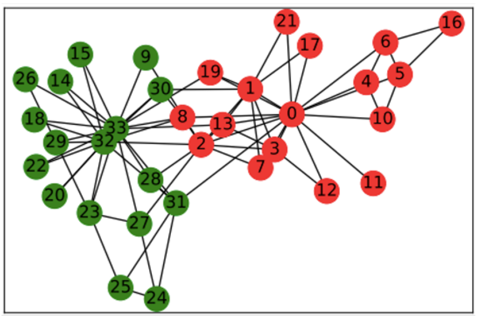


FIGURE 8. Graph of interactions between members of the Karate club as documented by Zachary

world while being more representative of our understanding of individuals holding various opinions for a range of topics.

In all the simulations for Case 1, Case 2 and Zachary's Karate Club, structural balance was observed. It is important to note however, that in the simulations for Case 2, the definition of structural balance is unclear. Since each entry in the weight matrix $w_{ij} \in \mathbb{R}^{2x2}$, which requires a new definition of structural balance. We consider this interesting future work. Regardless, even in Case 2, the structural balance was seen in $W \in \mathbb{R}^{nxn}$ when evaluated by considering W as $W \in \mathbb{R}^{2nx2n}$- since each entry $w_{ij} \in \mathbb{R}^{2x2}$.

V. CONCLUSION AND FUTURE WORK

In this paper, we extend and evaluate the co-evolution model proposed by Wang et al [3]. In [3], the adjacency matrix corresponding to a network of individuals W, captures the relationship between two individuals with a single real number. Since, in the real world, individuals hold multiple opinions on various topics, they have more complex relationships with those around them. Individuals may agree on certain issues while disagreeing on others and these complex interactions between individuals may shape their opinions and their relationship (social ties) in different ways. To incorporate this intuitive real world notion, we propose representing the relationship between to individuals by a $2x2$ matrix as opposed to a real number. Similarly, instead of individuals having opinions that are real numbers, we modify the opinions to be vectors which capture individuals' opinions on different issues.

The results from past work [3], show that when a network evolves in accordance with the co-evolution model, which is based on combining structural balance dynamics and opinion dynamics, structural balance is observed at the limit states and the magnitude of social ties and opinions goes to positive/negative infinity. In the real world however, this would mean limitless polarization and isolation between two communities. So, we propose to normalize the matrix of social ties W and the opinion vector of the individuals in the network at each iteration of the co-evolution model.

Our results are similar to those in [3] and structural balance is observed in all cases except Case 2 as described in Results section which is somewhat undefined. The results show polarization/harmony/oscillation of the opinions of individuals in a network while their social ties remain constant due to normalization. The normalization of the weight matrix allows for the system to evolve in a similar manner since the relative values of the social ties remain unchanged while preventing the ties of heading to infinity.

The scope for future work in this area remains strong as such models will find applications in several areas such as robot swarm control, behavioral economics, game theory and more pressing issues relating to tech policy, information warfare and adversarial manipulation in social networks.

Characterising structural balance for the weight matrix for higher dimensional cases remains an open area. Further, the co-evolution model requires rigorous analysis in the case of directed graphs which are still more representative of the real world. A comprehensive evaluation of the initial states of the social tie/weight matrix that lead to harmony/polarization or convergence is needed.

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Appendixes, if needed, appear before the acknowledgment.

ACKNOWLEDGMENT

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APPENDIX A REFERENCES

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