The Grassmannian

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Definition

Let V be an n-dimensional vector space, and fix an integer $d < n$

- The Grassmannian, denoted $Gr_{d,V}$ is the set of all d-dimensional vector subspaces of V
- This is a manifold and a variety

Some Simple Examples

For very small d and n, the Grassmannian is not very interesting, but it may still be enlightening to explore these examples in R^n

1. $Gr_{1,2}$ - All lines in a 2D space $\rightarrow \mathbb{P}$
2. $Gr_{1,3}$ - $\mathbb{P}^2$
3. $Gr_{2,3}$ - we can identify each plane through the origin with a unique perpendicular line that goes through the origin $\rightarrow \mathbb{P}^2$

The First Interesting Grassmannian

Let's spend some time exploring $Gr_{2,4}$, as it turns out this the first Grassmannian over Euclidean space that is not just a projective space.

- Consider the space of rank 2 (2×4) matrices with $A \sim B$ if $A = CB$ where $det(C) > 0$
- Let B be a (2×4) matrix. Let B_{ij} denote the minor from the ith and jth column. A simple computation shows

$$B_{12}B_{34} - B_{13}B_{24} + B_{14}B_{23} = 0$$

- Let $\Omega \subset \mathbb{P}^5$ be such that $(x, y, z, t, u, v) \in \Omega \iff xy - zt + uv = 0$. Define a map $f : M(2 \times 4) \to \Omega \subset \mathbb{P}^5$, $f(B) = (B_{12}, B_{34}, B_{13}, B_{24}, B_{14}B_{23})$. It can be shown this is a bijection

The First Interesting Grassmannian Continued...

Note for any element in Ω, and we can consider $(1, y, z, t, u, v)$ and note that the matrix

$$\begin{bmatrix} 1 & 0 & -v & -t \\ 0 & 1 & z & u \end{bmatrix}$$

will map to this under f, showing we really only need 4 parameters, i.e, the dimension is $k(n-k) = 2(4-2) = 4$ We hope to observe similar trends in the more general case

An Atlas for the Grassmannian

We will now show that $Gr_{k,V}$ is a smooth manifold of dimension $k(n-k)$.

- We identify linear subspaces of dimension k as maps from R^k to R^{n-k}. Let P be points such that k of its coordinates are nonzero, and Q be the subspace so that the other $n-k$ coordinates can be nonzero.
    1. $\rightarrow$: We note any element in this $k-\textit{dimensional}$ space can be written as the sum of an element in $p \in P$ and an element in $q \in Q$, so define a map that sends p to q
    2. $\leftarrow$: The graph of such a map A is a $k-\textit{dimensional}$ subspace defined by $\{x + Ax | x \in P \subset R^K\}$.
- Since such maps can be represented by a $(n-k) \times k$ matrix, this gives us what we need, since we can just consider the $(n-k)k$ entries as a vector in $R^{k(n-k)}$
- For now, I will neglect showing compatibility of charts

The Plucker Embedding

We have an embedding of $Gr_{k,V}$ into the projectivization of the exterior algebra of V, $P : Gr_{k,V} \to \mathbb{P}(\bigwedge^k V)$

- The map sends the basis of a subspace to the wedge product of the basis vectors
- can we break this down together?

For $V = C^n$, we get an embedding into projective space of dimension $\binom{n}{k} - 1$. Can we go over this and make this super clear?

- the canonical basis of $\bigwedge^k V$ is $\{e_{i_1} \wedge ... \wedge e_{i_d} | 1 \leq 1_1 < ... < i_d \leq n\}$, which has dimension $\binom{n}{k}$.

The Plucker Embedding Continued...

The embedding satisfies the plucker relations. We fix a basis of V, and take another basis for a subspace of dimension k. We can then consider the $k \times n$ matrix that gives the coordinates of this subbasis with respect to the basis of V. Then

- A plucker coordinate, denoted $W_{i_1,\ldots i_k}$ is the determinant of the minor formed by choosing those k columns

More Examples

1. For $k = 1$, we always get projective space of dimension $n - 1$. Any one dimensional subspace is spanned by a single vector, $a_1 e_1 + ... + a_n e_n$, yielding the matrix $(a_1, ..., a_n)$
2. We can now go back to our 2×4 example and do it a bit more formally. With a basis $\{e_1, e_2, e_3, e_4\}$ we have a wedge product basis of $\{e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4\}$. Then if a basis for a dimension 2 subspace is u_1, u_2, we write $u_1 = a_{11} e_1 + ... + a_{14} e_4, u_2 = a_{21} e_1 + ... + a_{24} e_4$, then $u_1 \wedge u_2 = \sum_{i=1}^{3} \sum_{j=2}^{4} A_{ij} e_i \wedge e_j$, where A_{ij} is the minor from the *ith* and *jth* column of $\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix}$, so the plucker coordinates are the tuple of these minors

The Grassmannian as a Variety

- A vector $v \in \bigwedge^n V$ is totally decomposable if there exist n LI $v_i \in V$ such that $v = v_i \wedge ... \wedge v_n$
- We identify the image of the Grassmanian under the Plucker map with the totally decomposable vectors of $\bigwedge^k V$
- Define a map $\phi_w : v \to \bigwedge^{k+1} V$ by $\phi_w(v) = w \wedge v$
- for each totally decomposable vector w, ϕ_w has rank $n - k$
- The map that sends w to ϕ_w is linear, so the entries of $\phi_w \in Hom(V, \bigwedge^{k+1} V)$ are homogeneous coordinates on $\mathbb{P}(\bigwedge^k V)$
- Then we can identify the image of the grassmannian as a subvariety defined by the vanishing of the minors of this matrix
- very unclear about this

Sources

- http://www.math.toronto.edu/mgualt/courses/18-367/docs/DiffGeomNotes-2.pdf
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