The Grassmannian

Sam Panitch
under the mentorship of Professor Chris Woodward and Marco Castronovo
supported by the Rutgers Math Department

June 11, 2020

DIMACS REU at Rutgers University
Let $V$ be an $n$-dimensional vector space, and fix an integer $d < n$

- The Grassmannian, denoted $Gr_{d,V}$ is the set of all $d$-dimensional vector subspaces of $V$
- This is a manifold and a variety
For very small $d$ and $n$, the Grassmannian is not very interesting, but it may still be enlightening to explore these examples in $\mathbb{R}^n$

1. $Gr_{1,2}$ - All lines in a 2D space $\rightarrow \mathbb{P}$
2. $Gr_{1,3}$ - $\mathbb{P}^2$
3. $Gr_{2,3}$ - we can identify each plane through the origin with a unique perpendicular line that goes through the origin $\rightarrow \mathbb{P}^2$
Let’s spend some time exploring $Gr_{2,4}$, as it turns out this the first Grassmannian over Euclidean space that is not just a projective space.

- Consider the space of rank 2 $(2 \times 4)$ matrices with $A \sim B$ if $A = CB$ where $\det(C) > 0$
- Let $B$ be a $(2 \times 4)$ matrix. Let $B_{ij}$ denote the minor from the $i$th and $j$th column. A simple computation shows

$$B_{12}B_{34} - B_{13}B_{24} + B_{14}B_{23} = 0$$

- Let $\Omega \subset \mathbb{P}^5$ be such that

$$(x, y, z, t, u, v) \in \Omega \iff xy - zt + uv = 0.$$ Define a map $f : M(2 \times 4) \to \Omega \subset \mathbb{P}^5$, $f(B) = (B_{12}, B_{34}, B_{13}, B_{24}, B_{14}B_{23})$. It can be shown this is a bijection.
Note for any element in $\Omega$, and we can consider $(1, y, z, t, u, v)$ and note that the matrix

$$
\begin{bmatrix}
1 & 0 & -v & -t \\
0 & 1 & z & u
\end{bmatrix}
$$

will map to this under $f$, showing we really only need 4 parameters, i.e, the dimension is $k(n - k) = 2(4 - 2) = 4$ We hope to observe similar trends in the more general case
We will now show that $Gr_k,\mathbb{V}$ is a smooth manifold of dimension $k(n - k)$.

- We identify linear subspaces of dimension $k$ as maps from $\mathbb{R}^k$ to $\mathbb{R}^{n-k}$. Let $P$ be points such that $k$ of its coordinates are nonzero, and $Q$ be the subspace so that the other $n - k$ coordinates can be nonzero.
  1. $\rightarrow$: We note any element in this $k$–dimensional space can be written as the sum of an element in $p \in P$ and an element in $q \in Q$, so define a map that sends $p$ to $q$.
  2. $\leftarrow$: The graph of such a map $A$ is a $k$–dimensional subspace defined by \( \{x + Ax | x \in P \subset \mathbb{R}^K\} \).

- Since such maps can be represented by a $(n - k) \times k$ matrix, this gives us what we need, since we can just consider the $(n - k)k$ entries as a vector in $\mathbb{R}^{k(n-k)}$.
- For now, I will neglect showing compatibility of charts.
The Plucker Embedding

We have an embedding of $Gr_{k,V}$ into the projectivization of the exterior algebra of $V$, $P : Gr_{k,V} \to \mathbb{P}(\bigwedge^k V)$

- The map sends the basis of a subspace to the wedge product of the basis vectors
- can we break this down together?

For $V = \mathbb{C}^n$, we get an embedding into projective space of dimension $(\binom{n}{k}) - 1$. Can we go over this and make this super clear?

- the canonical basis of $\bigwedge^k V$ is
  \[ \{ e_{i_1} \wedge ... \wedge e_{i_d} \mid 1 \leq i_1 < ... < i_d \leq n \} \], which has dimension $\binom{n}{k}$. 
The embedding satisfies the plucker relations. We fix a basis of $V$, and take another basis for a subspace of dimension $k$. We can then consider the $k \times n$ matrix that gives the coordinates of this subbasis with respect to the basis of $V$. Then

- A plucker coordinate, denoted $W_{i_1, \ldots, i_k}$ is the determinant of the minor formed by choosing those $k$ columns
1. For $k = 1$, we always get projective space of dimension $n - 1$. Any one dimensional subspace is spanned by a single vector, $a_1 e_1 + ... + a_n e_n$, yielding the matrix $(a_1, ..., a_n)$.

2. We can now go back to our $2 \times 4$ example and do it a bit more formally. With a basis $\{e_1, e_2, e_3, e_4\}$ we have a wedge product basis of $\{e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4\}$. Then if a basis for a dimension 2 subspace is $u_1, u_2$, we write $u_1 = a_{11} e_1 + ... + a_{14} e_4, u_2 = a_{21} e_1 + ... + a_{24} e_4$, then $u_1 \wedge u_2 = \sum_{i=1}^{3} \sum_{j=2}^{4} A_{ij} e_i \wedge e_j$, where $A_{ij}$ is the minor from the $ith$ and $jth$ column of \[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} \\
a_{21} & a_{22} & a_{23} & a_{24}
\end{bmatrix}
\], so the plucker coordinates are the tuple of these minors.
The Grassmannian as a Variety

- A vector $v \in \bigwedge^n V$ is totally decomposable if there exist $n$ LI $v_i \in V$ such that $v = v_1 \wedge \ldots \wedge v_n$
- We identify the image of the Grassmanian under the Plucker map with the totally decomposable vectors of $\bigwedge^k V$
- Define a map $\phi_w : v \rightarrow \bigwedge^{k+1} V$ by $\phi_w(v) = w \wedge v$
- for each totally decomposable vector $w$, $\phi_w$ has rank $n - k$
- The map that sends $w$ to $\phi_w$ is linear, so the entries of $\phi_w \in \text{Hom}(V, \bigwedge^{k+1} V)$ are homogeneous coordinates on $\mathbb{P}(\bigwedge^k V)$
- Then we can identify the image of the grassmannian as a subvariety defined by the vanishing of the minors of this matrix
- very unclear about this
Sources

- http://elib.mi.sanu.ac.rs/files/journals/tm/27/tm1428.pdf
- http://elib.mi.sanu.ac.rs/files/journals/tm/27/tm1428.pdf