Dimension of a Variety

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Preliminary Definitions

We first remember the following definitions. Let K be an algebraically closed field, and let K^n be n-dimensional affine space over K. Let $A \subset K^n$ be a variety

1. We define $I(A) = \{f \in K[x_1, x_2, ..., x_n] | f(a) = 0 \forall a \in A\}$
2. The coordinate ring of a variety is then $K[x_1, ..., x_n]/I(A)$
3. Given a ring R, we define the krull dimension to be the supremum of the length of all chains of prime ideals
4. The Zariski topology on a variety X has subvarieties of X as its closed sets

Dimension of an Affine Variety

- The dimension of a variety A is equal to the krull dimension of its coordinate ring

In short, this actually follows from a more general notion of dimension of a topological space. Given a topological space X, we can define its dimension to be the supremum of lengths of chains of irreducible closed subsets.

- The codimension of A in B is equal to the codimension of the prime ideal $I(A)$ in the coordinate ring of B

Proof (leaving out some details)

Let X be a subvariety of Y. Let $A(X)$ and $A(Y)$ denote their coordinate rings

1. We have $A(X) \cong A(Y)/I_Y(X)$, which is an integral domain iff $I_Y(X)$ is prime.
2. We have that a variety is irreducible iff its coordinate ring is an integral domain
3. Then, relative nullstellensatz restricts to a bijection between irreducible subvarieties of Y and prime ideals of $A(Y)$
4. Using the Zariski topology, this tells us that chains of closed subsets correspond to chains of prime ideals, which is exactly the definition of krull dimension

A simple motivating example

The set of polynomials that are 0 everywhere of course only contains the $0-polynomial$, so that $I(K^n)$ is the 0 ideal and its coordinate ring is $K[x_1, ..., x_n]$
This has dimension n, proving the dimension of K^n is n, as we would expect
Indeed we the chain of ideals $(x_1) \subset (x_1, x_n) \subset ... \subset (x_1, ..., x_n)$.
Unfortunately, the proof it is not greater than n is not at all trivial.

Properties of Dimension

- the dimension is always finite

Given two varieties A and B with dimension n and m respectively, we have

- $dim(A \times B) = n + m$
- if $A \subset B$, then $m = n + codim_B(A)$
- if f is an element of the coordinate ring of A, then every irreducible component of the variety generated by f has dimension $n - 1$

More Examples

- Consider $V(x_2 - x_1^2) \subset C^2$
  1. its coordinate ring is $C[x_1, x_2]/(x_2 - x_1^2) = C[x_1]$, which is an I.D. so that it is irreducible
  2. It has dimension one because its the zero locus of one polynomial in 2D affine space
- Consider $V(x_1 x_3, x_2 x_3)$ which looks like the union of a line and a plane
  1. This has dimension 2, and we can roughly see this by the inclusion of a point in a line in a plane

Sources

- https://www.mathematik.uni-kl.de/ gathmann/class/alggeom-2019/alggeom-2019.pdf