Mutations of Polynomials

Anna Antal and Sam Panitch
under the mentorship of Professor Chris Woodward and Marco Castronovo
supported by the Rutgers Math Department

June 1, 2020

DIMACS REU at Rutgers University

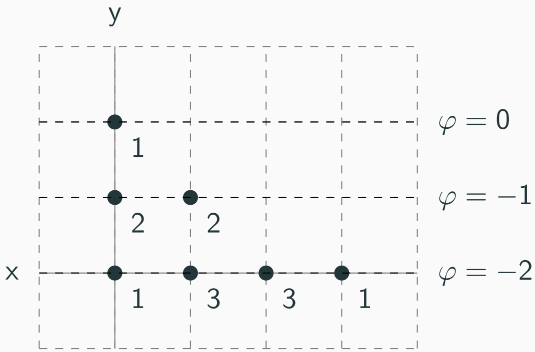
Notation and Definitions

- A lattice $L \cong \mathbb{Z}^2$ can be thought of as a coordinate grid
- An affine transformation $\varphi : L \to \mathbb{Z}$ is of the form $\varphi(v) = Av + b$
- The "linear part" of φ is $\varphi_o(v) = Av$

Mutation Data

- A mutation data is a pair (φ, h) where $\varphi : L \to \mathbb{Z}$ is a nonconstant affine transformation, and h is an element of the lattice and in the kernel of φ_o.
- We define the mutation associated to (φ, h) by $mut_{(\varphi,h)} : x^\ell \mapsto x^\ell h^{\varphi(\ell)}$

Example



Here we let $\varphi = x - 2$ (the height function minus two)

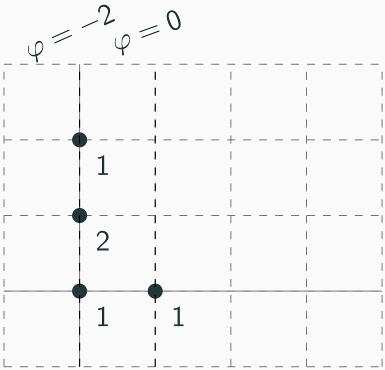
Thus, any polynomial in just x will be in the kernel of the linear part (x)

Choose $h = 1 + x$, and notice

$1+3x+3x^2+x^3 = (1+x)^3$

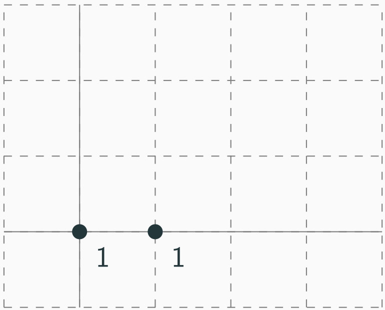
$2y + 2xy = 2y(1 + x)$

After first mutation



For our next mutation, we are free to choose any other affine *transformation*. We let $\varphi = 2y - 2$
Then, choose $h = 1 + y$, and notice $1 + 2y + y^2 = (1 + y)^2$

After second mutation

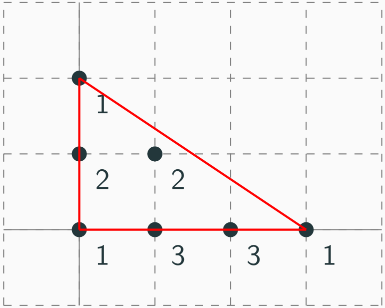


Notice we are left with the polynomial $1 + x$. From here, it is easy to reduce to a constant

0-Mutable

- x^{ℓ} is 0-mutable for all elements ℓ of the lattice
- $x^{\ell_1} + x^{\ell_2}$ is 0-mutable for all elements l_1, l_2 of the lattice where $\ell_1 - \ell_2$ is primitive
- Let f_1, f_2 be any polynomials of two variables. If they are 0-mutable, then so is $f_1 f_2$.
- Let f be a polynomial of two variables. If f is 0-mutable, then so is any mutation of f.

The Lattice Geometry of Polynomials



To any polynomial we can associate its convex hull, the smallest convex polygon containing its lattice points

Natural to ask whether or not the geometry of the convex hull affects

$k - mutability$

Questions to Explore

1. What is the appropriate definition of k-mutable?
   - if equivalent to a polynomial with $k+1$ terms?
2. How many equivalence classes of $k-mutable$ polynomials are there?
3. How can we determine whether or not a $k-mutable$ polynomial can actually be reduced further?
4. Connections to cluster algebras (Structures that generalize the notion of mutation)

Acknowledgements

- We thank Professor Chris Woodward and Marco Castronovo for their mentorship
- Additional gratitude goes out to the Rutgers Math Department for their support
- And finally, we thank you for listening!