Cluster Algebras: An Introduction

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May 28, 2020

DIMACS REU at Rutgers University

Section 1: Introduction

1. Cluster Algebra of rank n: A subfield of a field of rational functions in n variables
    - Requires data about a seed (n generators called cluster variables and an exchange matrix)
    - From the seed, use a process called mutation to obtain the rest of the cluster variables
2. Cluster: overlapping algebraically independent subsets that compose the cluster algebra
    - related to each other by birational transformations (so that coordinates are expressed rationally in terms of the others) of the form
      $$xx' = y^+M^+ + y^-M^-$$
    - Here, the Ms are monomials in the variables in the x cluster, and ys lie in a coefficient semifield

Questions on Introduction

- What is a coefficient semifield?
- These plus and minuses denote what exactly?
- Concise definition of ambient field?

Quivers

- Definition: a finite oriented graph with no loops nor oriented 2-cycles
- Quiver Mutation at Vertex k:
  1. for each subquiver $i \rightarrow k \rightarrow j$, add an arrow from i to j (unless both i and j are frozen)
  2. reverse all arrows with target or source k
  3. remove oriented two-cycles until unable to do so

A Quiver Mutation Example

Initial Quiver

$$\begin{tikzcd}
\boxed{c} & k \arrow[l] \arrow[d] & \boxed{d} \arrow[l] \\
a \arrow[u] \arrow[ur] & b \arrow[l] &
\end{tikzcd}$$

Step 1

$$\begin{tikzcd}
\boxed{c} & k \arrow[l] \arrow[d] & \boxed{d} \arrow[l] \arrow[dl] \\
a \arrow[u, shift left] \arrow[u, shift right] \arrow[ur] \arrow[r, shift right] & b \arrow[l, shift right] &
\end{tikzcd}$$

Step 2

$$\begin{tikzcd}
\boxed{c} \arrow[r] & k \arrow[r] \arrow[dl] & \boxed{d} \arrow[dl] \\
a \arrow[u, shift left] \arrow[u, shift right] \arrow[r, shift right] & b \arrow[l, shift right] \arrow[u] &
\end{tikzcd}$$

Step 3

$$\begin{tikzcd}
\boxed{c} \arrow[r] & k \arrow[r] \arrow[dl] & \boxed{d} \arrow[dl] \\
a \arrow[u, shift left] \arrow[u, shift right] & b \arrow[u] &
\end{tikzcd}$$

The Adjacency Matrix

- Signed Adjacency Matrix of a Quiver
  - label each vertex with a number from 1 to (# of vertices)
  - $b_{ij} = -b_{ji} = \ell$ where ℓ is the number of arrows from vertex i to j
  - Clearly skew-symmetric
- Adjacency Matrix under a mutation:

$$b'_{ij} = \begin{cases} -b_{ij} & i = k \text{ or } j = k \\ b_{ij} + b_{ik} b_{kj} & b_{ik} > 0 \text{ and } b_{kj} > 0 \\ b_{ij} - b_{ik} b_{kj} & b_{ik} < 0 \text{ and } b_{kj} < 0 \\ b_{ij} & \text{otherwise} \end{cases}$$

Seeds

Let $m \geq n$ and take F to be an ambient field of rational functions in n variables over $Q(x_{n+1}, ..., x_m)$. Take $k \in \{1, ..., n\}$.

- labeled seed: a pair $(\boldsymbol{x}, Q)$ where $\boldsymbol{x}$ forms a generating set for F and Q is a quiver with m vertices, the first n of which are mutable, the rest frozen
- extended cluster (of a seed): the $\boldsymbol{x}$ part
- cluster variables: the variables that are mutable
- frozen/coefficient variables: the variables that do not change

Seed Mutations

Define maps s and t on a quiver that take an arrow to its source and target respectively

- Seed Mutation: The mutation μ_k takes $(\boldsymbol{x}, Q) \to (\boldsymbol{x'}, \mu_k(Q))$ where $x' = x$ except at index k, where we have

  $$x'_k x_k = \prod_{\alpha \in Q, s(\alpha)=k} x_{t(\alpha)} + \prod_{\alpha \in Q, t(\alpha)=k} x_{s(\alpha)}$$

  Intuitively, this first product says find all the vertices that are hit by an arrow leaving vertex k, and multiply all of the corresponding elements in the extended cluster, while the second product says find all of the vertices that have arrows that lead into k and multiply all of the corresponding elements in the extended cluster

A simple example to do together

Let Q be the quiver on 2 vertices with an arrow from 1 to 2, and take our seed as $((x_1, x_2), Q)$. Let's mutate in the direction of 1 and then 2.

$$\begin{tikzcd} \overset{1}{\underset{x_1}{\bullet}} \arrow[r] & \overset{2}{\underset{x_2}{\bullet}} \arrow[r, Rightarrow, "\mu_1"] & \overset{1}{\bullet} & \overset{2}{\bullet} \arrow[l] \arrow[r, Rightarrow, "\mu_2"] & \overset{1}{\bullet} \arrow[r] & \overset{2}{\bullet} \end{tikzcd}$$

Our Not-Totally-General Definition of Cluster Algebra

Consider a regular n tree T_n with the n edges leaving each vertex labeled from $1, ..., n$.

- Cluster Pattern: an assignment of a labeled seed to each vertex such that the seeds connected by an edge k are related by a mutation in direction k. We denote the components of the extended cluster by $x_{i,t}$
- Take the union of the clusters of all of the seeds in the pattern, with cluster variables $x_{i,t}$

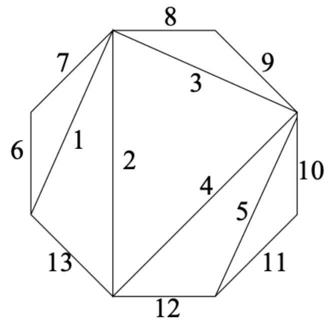
Now we can finally consider another definition of cluster algebra

- Cluster Algebra: The subalgebra generated by all of the cluster variables from above

Note this forms a subalgebra of the ambient field generated by all cluster variables. We generally write $\mathcal{A} = \mathcal{A}(\boldsymbol{x}, Q)$

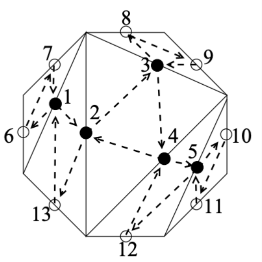
Triangulations of an n-gon

- We can build triangulations of an n-gon by adding $n - 3$ non-intersecting diagonals.
- Each diagonal is the diagonal of some quadrilateral, we can reach a new triangulation of the n-gon by swapping this diagonal for the other one of the same quadrilateral.



The Quiver Associated to a Triangulation

1. Place a mutable vertex at the midpoint of each diagonal, and a frozen vertex at the midpoint of each side
2. Take a triangle in the triangulation, the midpoints of its sides, and thus the corresponding vertices we placed there, determine a new triangle
3. Draw arrows in clockwise fashion around each of these new triangles



The Associated Cluster Algebra

- Take a triangulation of a d-gon, let $m = 2d - 3$, and let $n = d - 3$. Set $x = (x_1, ..., x_m)$. Then $(x, Q(T))$ is a labeled seed and it determines a cluster algebra $\mathcal{A} = \mathcal{A}(\boldsymbol{x}, Q)$
- A flip corresponds to a mutation in the quiver at the vertex on the edge we flipped
- The cluster algebras generated by two different triangulations of a d-gon are isomorphic, and any triangulation can be reached from another by a series of flips

The Grassmanian $\mathrm{Gr}_{2,d}$

- Points defined by full rank $2 \times m$ matrices with complex entries.
- Plucker coordinates, P_{ij}, denote the minor obtained from the ith and jth columns
- These Plucker coordinates can be related to triangulations if we label the vertices of the n-gon $1, ..., n$ and give the edge connecting vertex i to j the name P_{ij}
- It is easy to show they satisfy $P_{ik}P_{jl} = P_{ij}P_{kl} + P_{il}P_{jk}$

Questions

1. How to prove $Gr_{2,d} \simeq \mathbb{A}_{d-3}$?
2. Intuition for what $Gr_{2,d}$ actually is (I've taken very little algebraic geometry)
3. A cluster pattern is uniquely determined by an arbitrary seed, so how does taking the union of all clusters of the seeds in a pattern help? Aren't they all connected by mutations by definition?
4. What proofs for these quiver problems look like...

- tropical semifield: Let $Trop(u_1, ..., u_m)$ be the free multiplicative abelian group generated by the $u_i s$. Define $\bigoplus$ by

  $$\Pi_j u_j^{a_j} \bigoplus \Pi_j u_j^{b_j} = \Pi_j u_j^{min(a_j, b_j)}$$

  We call this group with this addition and its multiplication a tropical semifield.

Let $\mathcal{F}$ be an ambient field isomorphic to the field of rational functions in n variables with coefficients in QP (the group ring with coefficients in Q over the semifield we are using)

- labeled seed: a triple $(\boldsymbol{x}, \boldsymbol{y}, B)$ where
  1. $\boldsymbol{x}$ is an n tuple forming a free-generating set over QP
  2. $\boldsymbol{y}$ is an n tuple from P
  3. B is an $n \times n$ matrix that is skew-symmetrizable

Seed Mutations

We have $\mu_k(\boldsymbol{x}, \boldsymbol{y}, B) = (\boldsymbol{x}', \boldsymbol{y}', B')$ where

- $$b'_{ij} = \begin{cases} -b_{ij} & i = k \text{ or } j = k \\ b_{ij} + b_{ik} b_{kj} & b_{ik} > 0 \text{ and } b_{kj} > 0 \\ b_{ij} - b_{ik} b_{kj} & b_{ik} < 0 \text{ and } b_{kj} < 0 \\ b_{ij} & \text{otherwise} \end{cases}$$

  Note this is the same as before

- $$y'_j = \begin{cases} y_k^{-1} & j = k \\ y_j y_k^{[b_{kj}]_+} (y_k \bigoplus 1)^{-b_{kj}} & else \end{cases}$$

- $x'_j = x_j$ for $j \neq k$ and

  $$x'_k = \frac{y_k \Pi x_i^{[b_{ik}]_+} + \Pi x_i^{[-b_{ik}]_+}}{(y_k \bigoplus 1) x_k}$$

The Definition of a Cluster Algebra (again)

Consider a regular n tree T_n with the n edges leaving each vertex labeled from $1, ..., n$.

- Cluster Pattern: an assignment of a labeled seed to each vertex such that the seeds connected by an edge k are related by a mutation in direction k. We denote the components of the extended cluster by $x_t = (x_{1;t}, ...), y_t = (y_{1;t}, ...), B_t = (b_{ij}^t)$
- Take the union of the clusters of all of the seeds in the pattern
- Cluster Algebra: The subalgebra generated by all of the cluster variables from above

If we choose the semifield to be the tropical semifield, and if B is skew-symmetric, then this reduces to the definition from before

Cluster Algebra Fun Facts

1. any cluster variable can be expressed as a Laurent Polynomial in the variables of an arbitrary cluster
2. conjecture - the coefficients of these Laurent Polynomials are non-negative integer combinations of elements in the semifield

Final Loose Ends

- Why do we need a regular tree to define a Cluster Algebra?
- Any questions on your end?
- Thanks for listening!