Cluster Algebras: An Introduction

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Section 1: Introduction

1. Cluster Algebra of rank n: A subfield of a field of rational functions in n variables
   - Requires data about a seed (n generators called cluster variables and an exchange matrix)
   - From the seed, use a process called mutation to obtain the rest of the cluster variables

2. Cluster: overlapping algebraically independent subsets that compose the cluster algebra
   - related to each other by birational transformations (so that coordinates are expressed rationally in terms of the others) of the form
     \[ xx' = y^+ M^+ + y^- M^- \]
   - Here, the Ms are monomials in the variables in the x cluster, and ys lie in a coefficient semifield
Questions on Introduction

- What is a coefficient semifield?
- These plus and minuses denote what exactly?
- Concise definition of ambient field?
Quivers

• Definition: a finite oriented graph with no loops nor oriented 2-cycles
• Quiver Mutation at Vertex $k$:
  1. for each subquiver $i \rightarrow k \rightarrow j$, add an arrow from $i$ to $j$ (unless both $i$ and $j$ are frozen)
  2. reverse all arrows with target or source $k$
  3. remove oriented two-cycles until unable to do so
A Quiver Mutation Example

Initial Quiver

Step 1

Step 2

Step 3
The Adjacency Matrix

- Signed Adjacency Matrix of a Quiver
  - label each vertex with a number from 1 to (# of vertices)
  - $b_{ij} = -b_{ji} = \ell$ where $\ell$ is the number of arrows from vertex $i$ to $j$
  - Clearly skew-symmetric

- Adjacency Matrix under a mutation:

$$b'_{ij} = \begin{cases} 
-b_{ij} & i = k \text{ or } j = k \\
 b_{ij} + b_{ik} b_{kj} & b_{ik} > 0 \text{ and } b_{kj} > 0 \\
 b_{ij} - b_{ik} b_{kj} & b_{ik} < 0 \text{ and } b_{kj} < 0 \\
 b_{ij} & \text{otherwise}
\end{cases}$$
Let $m \geq n$ and take $F$ to be an ambient field of rational functions in $n$ variables over $Q(x_{n+1}, \ldots, x_m)$. Take $k \in \{1, \ldots, n\}$.

- labeled seed: a pair $(x, Q)$ where $x$ forms a generating set for $F$ and $Q$ is a quiver with $m$ vertices, the first $n$ of which are mutable, the rest frozen
- extended cluster (of a seed): the $x$ part
- cluster variables: the variables that are mutable
- frozen/coefficient variables: the variables that do not change
Seed Mutations

Define maps $s$ and $t$ on a quiver that take an arrow to its source and target respectively

- **Seed Mutation**: The mutation $\mu_k$ takes $(x, Q) \rightarrow (x', \mu_k(Q))$ where $x' = x$ except at index $k$, where we have

$$x'_k x_k = \prod_{\alpha \in Q, s(\alpha) = k} x_t(\alpha) + \prod_{\alpha \in Q, t(\alpha) = k} x_s(\alpha)$$

Intuitively, this first product says find all the vertices that are hit by an arrow leaving vertex $k$, and multiply all of the corresponding elements in the extended cluster, while the second product says find all of the vertices that have arrows that lead into $k$ and multiply all of the corresponding elements in the extended cluster.
A simple example to do together

Let $Q$ be the quiver on 2 vertices with an arrow from 1 to 2, and take our seed as $((x_1, x_2), Q)$. Let’s mutate in the direction of 1 and then 2.

1 \rightarrow 2 \quad \mu_1 \quad 1 \leftarrow 2 \quad \mu_2
Our Not-Totally-General Definition of Cluster Algebra

Consider a regular \( n \) tree \( T_n \) with the \( n \) edges leaving each vertex labeled from 1, ..., \( n \).

- Cluster Pattern: an assignment of a labeled seed to each vertex such that the seeds connected by an edge \( k \) are related by a mutation in direction \( k \). We denote the components of the extended cluster by \( x_{i,t} \).
- Take the union of the clusters of all of the seeds in the pattern, with cluster variables \( x_{i,t} \).

Now we can finally consider another definition of cluster algebra

- Cluster Algebra: The subalgebra generated by all of the cluster variables from above.

Note this forms a subalgebra of the ambient field generated by all cluster variables. We generally write \( \mathcal{A} = \mathcal{A}(x, Q) \).
We can build triangulations of an n-gon by adding \( n - 3 \) non-intersecting diagonals.

Each diagonal is the diagonal of some quadrilateral, we can reach a new triangulation of the n-gon by swapping this diagonal for the other one of the same quadrilateral.
The Quiver Associated to a Triangulation

1. Place a mutable vertex at the midpoint of each diagonal, and a frozen vertex at the midpoint of each side.
2. Take a triangle in the triangulation, the midpoints of its sides, and thus the corresponding vertices we placed there, determine a new triangle.
3. Draw arrows in clockwise fashion around each of these new triangles.
The Associated Cluster Algebra

- Take a triangulation of a $d$-gon, let $m = 2d - 3$, and let $n = d - 3$. Set $x = (x_1, ..., x_m)$. Then $(x, Q(T))$ is a labeled seed and it determines a cluster algebra $\mathcal{A} = \mathcal{A}(x, Q)$

- A flip corresponds to a mutation in the quiver at the vertex on the edge we flipped

- The cluster algebras generated by two different triangulations of a $d$-gon are isomorphic, and any triangulation can be reached from another by a series of flips
Points defined by full rank $2 \times m$ matrices with complex entries.

Plucker coordinates, $P_{ij}$, denote the minor obtained from the $i$th and $j$th columns.

These Plucker coordinates can be related to triangulations if we label the vertices of the $n$-gon $1, \ldots, n$ and give the edge connecting vertex $i$ to $j$ the name $P_{ij}$.

It is easy to show they satisfy $P_{ik}P_{jl} = P_{ij}P_{kl} + P_{il}P_{jk}$. 
1. How to prove $Gr_{2,d} \simeq \Delta_{d-3}$?

2. Intuition for what $Gr_{2,d}$ actually is (I’ve taken very little algebraic geometry)

3. A cluster pattern is uniquely determined by an arbitrary seed, so how does taking the union of all clusters of the seeds in a pattern help? Aren’t they all connected by mutations by definition?

4. What proofs for these quiver problems look like...
• tropical semifield: Let $\text{Trop}(u_1, \ldots, u_m)$ be the free multiplicative abelian group generated by the $u_i$s. Define $\oplus$ by

$$\prod_j u_j^{a_j} \oplus \prod_j u_j^{b_j} = \prod_j u_j^{\min(a_j, b_j)}$$

We call this group with this addition and its multiplication a tropical semifield.

Let $\mathcal{F}$ be an ambient field isomorphic to the field of rational functions in $n$ variables with coefficients in $\mathbb{Q}P$ (the group ring with coefficients in $\mathbb{Q}$ over the semifield we are using).

• labeled seed: a triple $(x, y, B)$ where
  1. $x$ is an $n$ tuple forming a free-generating set over $\mathbb{Q}P$
  2. $y$ is an $n$ tuple from $P$
  3. $B$ is an $n \times n$ matrix that is skew-symmetrizable
Seed Mutations

We have \( \mu_k(x, y, B) = (x', y', B') \) where

\[
b'_{ij} = \begin{cases} 
-b_{ij} & i = k \text{ or } j = k \\
b_{ij} + b_{ik} b_{kj} & b_{ik} > 0 \text{ and } b_{kj} > 0 \\
b_{ij} - b_{ik} b_{kj} & b_{ik} < 0 \text{ and } b_{kj} < 0 \\
b_{ij} & \text{otherwise}
\end{cases}
\]

Note this is the same as before

\[
y_j' = \begin{cases} 
y_k^{-1} & j = k \\
y_j y_k^{[b_{kj}]} + (y_k \oplus 1)^{-b_{kj}} & \text{else}
\end{cases}
\]

\( x'_j = x_j \) for \( j \neq k \) and

\[
x'_k = \frac{y_k \prod_{i} x_i^{[b_{ik}]} + \prod_{i} x_i^{[-b_{ik}]} + (y_k \oplus 1)x_k}{(y_k \oplus 1)x_k}
\]
Consider a regular $n$ tree $T_n$ with the $n$ edges leaving each vertex labeled from $1, \ldots, n$.

- Cluster Pattern: an assignment of a labeled seed to each vertex such that the seeds connected by an edge $k$ are related by a mutation in direction $k$. We denote the components of the extended cluster by $x_t = (x_{1;t}, \ldots), y_t = (y_{1;t}, \ldots), B_t = (b_{ij}^t)$.

- Take the union of the clusters of all of the seeds in the pattern.

- Cluster Algebra: The subalgebra generated by all of the cluster variables from above.

If we choose the semifield to be the tropical semifield, and if $B$ is skew-symmetric, then this reduces to the definition from before.
Cluster Algebra Fun Facts

1. any cluster variable can be expressed as a Laurent Polynomial in the variables of an arbitrary cluster
2. conjecture - the coefficients of these Laurent Polynomials are non-negative integer combinations of elements in the semifield
Final Loose Ends

- Why do we need a regular tree to define a Cluster Algebra?
- Any questions on your end?
- Thanks for listening!