Characterizing *LEF* Groups

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Definition

A group G is said to be locally embeddable into finite groups (LEF) if for every finite subset $F \subseteq G$, there is an injection $\varphi : F \to H$ into a finite group H such that if $x, y, xy \in F$, then

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Nonexample

The Baumslag-Solitar group $BS(2,3) = \langle\, a, b \mid a^{-1}b^2a = b^3 \,\rangle$ is not *LEF*.

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If G is a countable group, then G is LEF if and only if G embeds into the reduced product P_0.

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The *LEF* group $\mathrm{SL}(3, \mathbb{Q})$ does not embed in $P = \prod_{n \in \mathbb{N}^+} S_n$.

Our work

Remark

Since the reduced product P_0 has cardinality $2^{\aleph_0}$, it is natural to ask whether the above characterization can be extended to the uncountable groups G such that $\aleph_0 < |G| \leq 2^{\aleph_0}$.

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Using model-theoretic techniques, we have proved:

Theorem (Stetson-Thomas)

If G is a group with $|G| = \aleph_1$, then G is LEF if and only if G embeds into the reduced product P_0.

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This solves our problem if *CH* holds; i.e. if $2^{\aleph_0} = \aleph_1$.

Question

What about if $2^{\aleph_0} > \aleph_1$?

Our work continues ...

Theorem (Stetson-Thomas)

It is consistent with ZFC that $2^{\aleph_0}$ *can be arbitarily large and whenever* G *is a group with* $\aleph_0 < |G| < 2^{\aleph_0}$, *then* G *is LEF if and only if* G *embeds into the reduced product* P_0.

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Theorem (Stetson-Thomas)

It is consistent with ZFC that $2^{\aleph_0}$ can be arbitarily large and there exists an LEF G with $|G| = \aleph_2$ such that G does not embed into the reduced product P_0.

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Remark

Thus *ZFC* neither proves nor disproves our characterization for groups of cardinality $\aleph_2$.

An Open Question

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But is it also consistent that our characterization is valid for groups of cardinality $2^{\aleph_0} > \aleph_1$?

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- We conjecture that this is indeed consistent.

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Question

But is it also consistent that our characterization is valid for groups of cardinality $2^{\aleph_0} > \aleph_1$?

Remark

- We conjecture that this is indeed consistent.
- However, it is clear that the proof will require more sophisticated forcing techniques.

Questions & Support

Any questions?

Thank you for your attention.

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