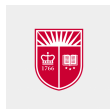


Characterizing *LEF* Groups

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21 July 2023



Definition

A group G is said to be **locally embeddable into finite groups** (LEF) if for every finite subset $F \subseteq G$, there is an injection $\varphi : F \rightarrow H$ into a finite group H such that if $x, y, xy \in F$, then

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Nonexample

The Baumslag-Solitar group $BS(2, 3) = \langle a, b \mid a^{-1}b^2a = b^3 \rangle$ is not *LEF*.

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Theorem (Folklore)

If G is a countable group, then G is LEF if and only if G embeds into the reduced product P_0 .

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Why factor by $N = \bigoplus_{n \in \mathbb{N}^+} S_n$?

The LEF group $\mathrm{SL}(3, \mathbb{Q})$ does **not** embed in $P = \prod_{n \in \mathbb{N}^+} S_n$.

Remark

Since the reduced product P_0 has cardinality 2^{\aleph_0} , it is natural to ask whether the above characterization can be extended to the uncountable groups G such that $\aleph_0 < |G| \leq 2^{\aleph_0}$.

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Using model-theoretic techniques, we have proved:

Theorem (Stetson-Thomas)

If G is a group with $|G| = \aleph_1$, then G is LEF if and only if G embeds into the reduced product P_0 .

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This solves our problem if *CH* holds; i.e. if $2^{\aleph_0} = \aleph_1$.

Question

What about if $2^{\aleph_0} > \aleph_1$?

Theorem (Stetson-Thomas)

It is consistent with ZFC that 2^{\aleph_0} can be arbitrarily large and whenever G is a group with $\aleph_0 < |G| < 2^{\aleph_0}$, then G is LEF if and only if G embeds into the reduced product P_0 .

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Theorem (Stetson-Thomas)

*It is consistent with ZFC that 2^{\aleph_0} can be arbitrarily large and there exists an LEF G with $|G| = \aleph_2$ such that G does **not** embed into the reduced product P_0 .*

Our work continues ...

Theorem (Stetson-Thomas)

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Theorem (Stetson-Thomas)

*It is consistent with ZFC that 2^{\aleph_0} can be arbitrarily large and there exists an LEF G with $|G| = \aleph_2$ such that G does **not** embed into the reduced product P_0 .*

Remark

Thus ZFC neither proves nor disproves our characterization for groups of cardinality \aleph_2 .

An Open Question

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But is it also consistent that our characterization is valid for groups of cardinality $2^{\aleph_0} > \aleph_1$?

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- We conjecture that this is indeed consistent.

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But is it also consistent that our characterization is valid for groups of cardinality $2^{\aleph_0} > \aleph_1$?

Remark

- We conjecture that this is indeed consistent.
- However, it is clear that the proof will require more sophisticated forcing techniques.

Any questions?

Thank you for your attention.

On the DIMACS REU side, at least, this work is being supported by the Rutgers Department of Mathematics and NSF Grant DMS-2019396.