LEF Groups

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LEF Groups

Definition

A group G is said to be locally embeddable into finite groups if for every finite subset $F \subseteq G$, there is an injection $\varphi : F \to H$ into a finite group H such that if x, y, $xy \in F$, then

$$\varphi(xy) = \varphi(x)\varphi(y).$$

In this case, we say that G is an *LEF* group.

Example

It can be shown (with considerable effort) that the general linear group $\mathrm{GL}(n, \mathbb{R})$ is an *LEF* group.

Characterizing *LEF* Groups

- In the literature, *LEF* groups are characterized as the groups which embed in a suitable ultraproduct of the finite symmetric groups S_n.
- Unfortunately, the ultraproduct construction makes use of a nonprincipal ultrafilter; and these cannot be proved to exist without the Axiom of Choice.
- In preliminary work, we have found a more concrete characterization of the countable *LEF* groups.

The Reduced Product

Definition

- Let $P = \prod_{n \in \mathbb{N}^+} S_n$ be the full direct product of the finite symmetric groups S_n.
- Let N be the normal subgroup of those elements $(\pi_n) \in P$ such that $\pi_n = 1$ for all but finitely many $n \in \mathbb{N}^+$.
- Then the reduced product of the S_n is the quotient P/N.

Theorem (Stetson-Thomas)

If G is a countable group, then the following are equivalent:

(i) *G is an LEF group.*

(ii) *G embeds into the reduced product P/N.*

The Research Problem

Remark

Since the reduced product P/N has cardinality $2^{\aleph_0}$, it is natural to ask whether the above characterization can be extended to the groups G such that $|G| \leq 2^{\aleph_0}$.

Research Problem

Is it true that if G is a group such that $|G| \leq 2^{\aleph_0}$, then the following are equivalent:

(i) *G is an LEF group.*

(ii) *G embeds into the reduced product P/N.*

Remark

We expect that this statement will turn out to be independent of the axioms *ZFC* of set theory.

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