LEF Groups

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A group $G$ is said to be locally embeddable into finite groups if for every finite subset $F \subseteq G$, there is an injection $\varphi : F \to H$ into a finite group $H$ such that if $x, y, xy \in F$, then

$$\varphi(xy) = \varphi(x)\varphi(y).$$

In this case, we say that $G$ is an LEF group.

It can be shown (with considerable effort) that the general linear group $\text{GL}(n, \mathbb{R})$ is an LEF group.
In the literature, \emph{LEF} groups are characterized as the groups which embed in a suitable ultraproduct of the finite symmetric groups $S_n$.

Unfortunately, the ultraproduct construction makes use of a nonprincipal ultrafilter; and these cannot be proved to exist without the Axiom of Choice.

In preliminary work, we have found a more concrete characterization of the countable \emph{LEF} groups.
The Reduced Product

Definition

Let $P = \prod_{n \in \mathbb{N}^+} S_n$ be the full direct product of the finite symmetric groups $S_n$.

Let $N$ be the normal subgroup of those elements $(\pi_n) \in P$ such that $\pi_n = 1$ for all but finitely many $n \in \mathbb{N}^+$.

Then the reduced product of the $S_n$ is the quotient $P/N$.

Theorem (Stetson-Thomas)

If $G$ is a countable group, then the following are equivalent:

(i) $G$ is an LEF group.

(ii) $G$ embeds into the reduced product $P/N$. 
The Research Problem

Remark
Since the reduced product $P/N$ has cardinality $2^{\aleph_0}$, it is natural to ask whether the above characterization can be extended to the groups $G$ such that $|G| \leq 2^{\aleph_0}$.

Research Problem

Is it true that if $G$ is a group such that $|G| \leq 2^{\aleph_0}$, then the following are equivalent:

(i) $G$ is an LEF group.
(ii) $G$ embeds into the reduced product $P/N$.

Remark
We expect that this statement will turn out to be independent of the axioms ZFC of set theory.
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