THE CHROMATIC POLYNOMIAL OF A GRAPH

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Project on Pure Math: Geometry and Combinatorics of Matroids
The Chromatic Polynomial

- **Proper Coloring:**
  - Two vertex on the same edge must have different colors

- **Definition of Chromatic Polynomial:**
  - The number of ways to color the vertices of a graph, $G$, using $q$ colors so that the endpoints of every edge have different colors.
Deletion-Contraction Relation

\[ \chi_G(q) = \chi_{G\setminus e}(q) - \chi_{G/e}(q) \]

G\setminus e – deletion of an edge, e, from the graph, G
G/e - contraction of an edge from the graph
Example of $C_4$ Graph

\[ q(q - 1)^3 - q(q - 1)(q - 2) = q^4 - 4q^3 + 6q^2 - 3q \]
\[ q(q - 1)^2 + q(q - 1)(q - 2)^2 = q^4 - 4q^3 + 6q^2 - 3q \]
A breakdown:

\[
q^4 - 4q^3 + 6q^2 - 3q
\]
Chromatic Polynomial Of Cycles:

\[ n = 1 \quad \chi(q) = 0 \]

\[ n = 2 \quad q \quad (q - 1) = q(q - 1) \]

\[ n = 3 \quad q \quad (q - 2) \quad (q - 1) = q(q - 1)(q - 2) \]
\[
\chi(q) = q(q - 1)^2 + q(q - 1)(q - 2)^2
\]

\[
\chi(q) = q(q - 1)^4 - (q(q - 1)^2 + (q(q - 1)(q - 2)^2))
\]

\[
\chi(q) = q(q - 1)^5 - (q(q - 1)^4 - (q(q - 1)^2 + (q(q - 1)(q - 2)^2)))
\]
General Solution for a Cycle:

\[ \chi_{c_n}(q) = \chi_{c_n}\text{e} - \chi_{c_{n-1}}(q) \]
Trees:

\[ \chi(q) = q(q - 1)^n \]

General Solution for Trees:

\[ \chi(q) = q(q - 1)^{n-1} \]
Open Wheels:

\[ n = 3 \]

\[ q \]

\[ q - 2 \]

\[ q - 1 \]

\[ q(q - 1)(q - 2) \]

\[ n = 4 \]

\[ q - 2 \]

\[ q - 2 \]

\[ q - 1 \]

\[ q \]

\[ q(q - 1)(q - 2)^2 \]

\[ \chi(q) = q(q - 1)(q - 2)^{n-2} \]

\[ n = 5 \]

\[ q - 2 \]

\[ q - 2 \]

\[ q - 1 \]

\[ q \]

\[ q - 2 \]

\[ q(q - 1)(q - 2)^3 \]
Wheels:

$n = 4$

$q(q - 1)(q - 2)(q - 3)$

$n = 5$

$q(q - 1)(q - 2)(q - 3)^2 + q(q - 1)(q - 2)^2$

General Solution: $q[(q - 2)^{n-1} + (-1)^{n-1}(q - 2)]$
Current Work:

- Working on finding the Chromatic Polynomial of a Matroid! That is a little more complicated because of the rules that apply to matroids.
- Also I’m working on the chromatic polynomial of 3-dimensional graphs.
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Citations: