A matrix sequence \( \{\Gamma(A^m)\}_{m=1}^{\infty} \) might converge even if the matrix \( A \) is not primitive

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**ARTICLE INFO**

**Article history:**
Received 29 May 2012
Accepted 15 October 2012
Available online 1 December 2012
Submitted by R.A. Brualdi

**AMS classification:**
05C20
05C50

**Keywords:**
Irreducible Boolean \((0, 1)\)-matrix
Powers of Boolean \((0, 1)\)-matrices
Competition graph
Graph sequence
Powers of digraphs

**ABSTRACT**

It is well-known that, for an irreducible Boolean \((0, 1)\)-matrix \( A \), the matrix sequence \( \{A^m\}_{m=1}^{\infty} \) converges if and only if \( A \) is primitive. In this paper, we introduce an operation \( \Gamma \) on the set of Boolean \((0, 1)\)-matrices such that a matrix sequence \( \{\Gamma(A^m)\}_{m=1}^{\infty} \) might converge even if the matrix \( A \) is not primitive. Given a Boolean \((0, 1)\)-matrix \( A \), we define a matrix \( \Gamma(A) \) so that the \((i, j)\)-entry of \( \Gamma(A) \) equals 0 if for \( i \neq j \), the inner product of the \( i \)th row and \( j \)th row of \( A \) is 0 and equals 1 otherwise.

The aim of this paper is to study the convergence of \( \{\Gamma(A^m)\}_{m=1}^{\infty} \) for a Boolean \((0, 1)\)-matrix \( A \) whose digraph has at most two strong components. We show that \( \{\Gamma(A^m)\}_{m=1}^{\infty} \) converges to a very special type of matrix as \( m \) increases if \( A \) is an irreducible Boolean matrix. Furthermore, we completely characterize a Boolean \((0, 1)\)-matrix \( A \) whose digraph has exactly two strongly connected components and for which \( \{\Gamma(A^m)\}_{m=1}^{\infty} \) converges, and find the limit of \( \{\Gamma(A^m)\}_{m=1}^{\infty} \) in terms of its digraph when it converges. We derive these results in terms of the competition graph of the digraph of \( A \).

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1. Introduction

The focus of this paper is a problem about the convergence of a certain sequence of Boolean \((0, 1)\)-matrices or equivalently the convergence of the \( m \)-step competition graphs of certain digraphs.

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1 This work was supported by National Research Foundation of Korea Grant funded by the Korean Government (NRF-2011-357-C00004).

2 This work was supported by the National Research Foundation of Korea (NRF) Grant funded by the Korea Government (MEST) (No. 2011-0005188).

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Throughout the paper, we will state definitions, facts, and theorems (where appropriate) in terms of both Boolean \((0, 1)\)-matrices and competition graphs.

As a matter of fact, for a matrix \(A\) in \(B_n\), we define a matrix \(\Gamma(A) = (\gamma_{ij}) \in B_n\) by

\[
\gamma_{ij} = \begin{cases} 
0 & \text{if } i = j; \\
0 & \text{if } i \neq j \text{ and the inner product of row } i \text{ and row } j \text{ of } A \text{ is 0}; \\
1 & \text{if } i \neq j \text{ and the inner product of row } i \text{ and row } j \text{ of } A \text{ is not 0}.
\end{cases}
\]

As a matter of fact, for a matrix \(A \in B_n\), \(\Gamma(A)\) is the adjacency matrix of the competition graph of the digraph \(D\) of \(A\). Given a matrix \(A\) in \(B_n\), there exists a unique digraph whose adjacency matrix is \(A\). We call such a digraph the digraph of \(A\) and denote it by \(D(A)\).

Given a digraph \(D\), the 
**competition graph** \(C(D)\) of \(D\) has the same vertex set as \(D\) and has an edge between vertices \(u\) and \(v\) if and only if there exists a common prey of \(u\) and \(v\) in \(D\). If \((u, v)\) is an arc of a digraph \(D\), then we call \(v\) a prey of \(u\) (in \(D\)) and call \(u\) a predator of \(v\) (in \(D\)). A graph \(G\) is called the 
**row graph** of a matrix \(M\) if the rows of \(M\) are the vertices of \(G\), and two vertices are adjacent in \(G\) if and only if their corresponding rows have a nonzero entry in the same column of \(M\). This notion was studied by Greenberg et al. [6]. As noted in [6], the competition graph of a digraph \(D\) is the row graph of its adjacency matrix. Thus it can easily be checked that the adjacency matrix of the competition graph of a digraph \(D\) is \(\Gamma(A)\) where \(A\) is the adjacency matrix of \(D\).

The notion of competition graph is due to Cohen [5] and has arisen from ecology. Competition graphs also have applications in coding, radio transmission, and modeling of complex economic systems. (See [13, 14] for a summary of these applications.)

The greatest common divisor of all lengths of directed cycles in a nontrivial strongly connected digraph \(D\) is called the 
**index of imprimitivity** of \(D\). A digraph \(D\) is said to be primitive if \(D\) is strongly connected and has the index of imprimitivity 1. Let \(A\) be a matrix in \(B_n\). If \(D(A)\) is strongly connected, then we say \(A\) is irreducible. We call the index of imprimitivity of \(D(A)\) the 
**index of imprimitivity** of \(A\), when \(A\) is irreducible. If \(D(A)\) is primitive, then we say that \(A\) is primitive. For undefined terms in the following, the reader is referred to [2].

It is well-known that for an irreducible matrix \(A\) in \(B_n\), the matrix sequence \(\{A^m\}_{m=1}^{\infty}\) converges if and only if \(A\) is primitive. Yet, a matrix sequence \(\{\Gamma(A^m)\}_{m=1}^{\infty}\) might converge even if the matrix \(A\) is not primitive. For example, the \(m\)th power of the matrix \(A\) given in Fig. 1 does not converge as \(m\) increases since it is not primitive. However, the \(\{\Gamma(A^m)\}_{m=1}^{\infty}\) converges to \(A'\) since \(\Gamma(A^m) = A'\) for any positive integer \(m\).

In this paper, we study the convergence of \(\{\Gamma(A^m)\}_{m=1}^{\infty}\) for a matrix \(A \in B_n\) whose digraph has at most two strongly connected components. We show that \(\{\Gamma(A^m)\}_{m=1}^{\infty}\) converges as \(m\) increases for any irreducible Boolean matrix \(A\) and its limit is a block diagonal matrix each of whose blocks consists of all 1s up to conjugation by simultaneous permutation of rows and columns. From now on, we call such a matrix 
**J block diagonal** (for short JBD) matrix (where \(J\) means a matrix with all 1s). Furthermore, we completely characterize a matrix \(A \in B_n\) whose digraph has exactly two strongly connected components and for which \(\{\Gamma(A^m)\}_{m=1}^{\infty}\) converges, and find the limit of \(\{\Gamma(A^m)\}_{m=1}^{\infty}\) in terms of its digraph when it converges. We derive these facts in terms of the competition graph of the digraph of \(A\).
Given a digraph $D$ and a positive integer $m$, a vertex $y$ is an $m$-step prey of a vertex $x$ if and only if there exists a directed walk from $x$ to $y$ of length $m$. Given a digraph $D$ and a positive integer $m$, the digraph $D^m$ has the vertex set same as $D$ and has an arc $(u, v)$ if and only if $u$ is an $m$-step prey of $v$. It is well-known that a digraph $D$ is primitive if and only if $D^m$ equals the digraph which has all possible arcs for any $m \geq N$ for some positive integer $N$ (we call the smallest such integer $N$ the exponent of $D$).

Motivated by this, we say that a graph sequence \{G\}_n=1 is well-known that a digraph $D$ is primitive if and only if $D^m$ has only complete components as $m$ increases if $D$ is strongly connected, completely characterize a digraph $D$ with exactly two strong components for which $(D^m)_m=1$ converges, and find the limit of $(D^m)_m=1$ when $(D^m)_m=1$ converges.

Given a positive integer $m$, the $m$-step competition graph of a digraph $D$, denoted by $C^m(D)$, has the same vertex set as $D$ and has an edge between vertices $u$ and $v$ if and only if there exists an $m$-step common prey of $u$ and $v$. The notion of $m$-step competition graph is introduced by Cho et al. \[4\] and one of the important variations (see the survey articles by Kim \[10\] and Lundgren \[12\] for the variations which have been defined and studied by many authors since Cohen introduced the notion of competition). Since its introduction, it has been extensively studied (see for example \[1,3,7–9,11,16\]).

Cho and Kim \[3\] showed that for any digraph $D$ and a positive integer $m$, $C^m(D) = C(D^m)$. Thus the limit of the graph sequence $(C(D^m))_m=1$ if it exists, is the same as that of the graph sequence $(C(D^m))_m=1$. Consequently studying the graph sequence $(C(D^m))_m=1$ is actually studying the sequence of $m$-step competition graphs of $D$.

2. $(\Gamma(A^m))_m=1$ for an irreducible matrix $A \in B_n$

In this section, we study the convergence and the limit graph of $(\Gamma(A^m))_m=1$ when $A$ is an irreducible matrix in $A \in B_n$. As mentioned previously, $\Gamma(A^m)$ corresponds to the competition graph of $D^m$ where $D$ is the digraph of $A$ for a positive integer $m$.

We start with the following observation.

**Theorem 2.1.** If a digraph $D$ is trivial or each vertex of $D$ has an out-neighbor, then $(C(D^m))_m=1$ converges.

**Proof.** The proposition immediately holds for a trivial digraph. Let $D$ be a nontrivial digraph such that each vertex has an out-neighbor. To show that $E(C(D^m)) \subseteq E(C(D^m+1))$ for any integer $m$, take an edge $uv$ in $E(C(D^m))$ for some positive integer $m$. Then there exists a vertex $z$ in $D$ such that $(u,z)$ and $(v,z)$ are arcs of $D^m$ for some vertex $z$. In $D$, $z$ is a common $m$-step prey of $u$ and $v$. By the assumption on $D$, there exists a vertex $x$ such that $(z,x) \in A(D)$. Then $x$ is a common $(m+1)$-step prey of $u$ and $v$ in $D$, that is, $x$ is a common prey of $u$ and $v$ in $D^m+1$. Therefore $u$ and $v$ are adjacent in $C(D^m+1)$. Thus we have shown that $E(C(D^m)) \subseteq E(C(D^m+1))$ for any integer $m$. Since the competition graph of a digraph is defined to be simple, $E(C(D^m)) \subseteq E(K_n)$ for each $m$ where $n = |V(D)|$. Therefore, it can easily be checked that there exists an integer $N$ such that for any $n \geq N$, $C(D^n) = C(D^N)$, which implies that $(C(D^m))_m=1$ converges. \[\square\]

Theorem 2.1 is translated into the matrix version as follows:

**Corollary 2.2.** If a Boolean $(0,1)$-matrix $A$ has order 1 or has at least one 1 in each row, then $(\Gamma(A^m))_m=1$ converges.

It is known that if $\kappa$ is the index of imprimitivity of a digraph $D$, then $D$ has an ordered partition \{$U_1, U_2, \ldots, U_\kappa$\} of $V(D)$ such that $U_{\kappa+1} = U_1$ and each arc of $D$ issues from $U_j$ and enters $U_{j+1}$ for some $j = 1, 2, \ldots, \kappa$. The sets $U_1, U_2, \ldots, U_\kappa$ are called the sets of imprimitivity of $D$. 
If $D$ is a trivial digraph, then the index of imprimitivity of $D$ is undefined. Given a strongly connected digraph $D$, we define $\kappa(D)$ as 1 if $D$ is trivial and as the index of imprimitivity of $D$ otherwise. In addition, if $D$ is a trivial digraph, then we denote the vertex set by $U_1$ and call it ‘the sets of imprimitivity of $D$’.

If a digraph $D$ is strongly connected, then the graph sequence $\{C(D^m)\}_{m=1}^{\infty}$ converges by Theorem 2.1. Then it is natural to ask: What is the limit of the sequence? In the rest of this section, we answer the question by showing that the limit of $\{C(D^m)\}_{m=1}^{\infty}$ is the union of exactly $\kappa(D)$ complete components.

**Theorem 2.3.** If $D$ is strongly connected, then the limit of $\{C(D^m)\}_{m=1}^{\infty}$ is the disjoint union of complete graphs whose vertex sets are the sets of imprimitivity of $D$.

The above theorem immediately implies the following corollary.

**Corollary 2.4.** If $A$ is an irreducible $(0, 1)$-Boolean matrix, then the limit of $\{\Gamma(A^m)\}_{m=1}^{\infty}$ is transformable into a block diagonal matrix by simultaneous permutations of their lines in which each block is in the form of $J_i$, where $J_i$ is a matrix with all elements 1 and $i$ runs from 1 to the index of imprimitivity of $A$.

The following is a well-known result related to the index of imprimitivity of a digraph.

**Theorem 2.5 [2, Theorem 3.4.5].** Let $D$ be a nontrivial strongly connected digraph of order $n$ with the index of imprimitivity $\kappa$, and $A$ be the adjacency matrix of $D$. Then there exists a permutation matrix $P$ of order $n$ such that

$$PA^\ell P^T = \begin{pmatrix}
A_1 & O & \cdots & O \\
O & A_2 & \cdots & O \\
\vdots & \vdots & \ddots & \vdots \\
O & O & \cdots & A_r
\end{pmatrix},$$

where $r = \gcd(\kappa, \ell)$ and each of $A_1, A_2, \ldots, A_r$ is an irreducible matrix with the index of imprimitivity $\frac{\ell}{r}$.

If a matrix $A$ in $B_n$ is primitive, then there exists a positive integer $m$ such that the $m$th power $A^m$ has only positive entries, and such smallest integer $m$ is called the exponent of $A$, which is denoted by $\exp(A)$.

Suppose that $D$ is a nontrivial strongly connected digraph. Let $A$ be the adjacency matrix of $D$ and $\kappa(D) = \kappa$. By Theorem 2.5, there exists a permutation matrix $P$ such that

$$PA^\kappa P^T = \begin{pmatrix}
A_1 & O & \cdots & O \\
O & A_2 & \cdots & O \\
\vdots & \vdots & \ddots & \vdots \\
O & O & \cdots & A_\kappa
\end{pmatrix},$$

where $A_i$ is primitive for any $1 \leq i \leq \kappa$. Let

$$M = \kappa \cdot \max\{\exp(A_i) \mid 1 \leq i \leq \kappa\}. \quad (2)$$

For simplicity, let $E = \max\{\exp(A_i) \mid 1 \leq i \leq \kappa\}$. Take a positive integer $s$. Since $PA^sM P^T = PA^s \cdot E P^T = (PA^\kappa P^T)^s E$.
PA^M P^T = \begin{pmatrix}
A_1^{SE} & 0 & \cdots & 0 \\
0 & A_2^{SE} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_k^{SE}
\end{pmatrix}

(3)

by (1). Since \( sE \geq \exp(A_i) \), any entry of \( A_i^{SE} \) is positive for each \( 1 \leq i \leq \kappa \). Therefore \( D(A_i^{SE}) \) is a digraph with all possible arcs for each \( 1 \leq i \leq \kappa \), and so \( D(A^{SM}) \) is a disjoint union of digraphs with all possible arcs, \( D(A_1^{SE}), D(A_2^{SE}), \ldots, D(A_k^{SE}) \).

Now we obtain the following lemma:

**Lemma 2.6.** Let \( D \) be a strongly connected digraph. Then there exists a positive integer \( M \) such that \( C(D^{SM}) \) has exactly \( \kappa(D) \) components all of which are complete for each positive integer \( s \).

**Proof.** Let \( \kappa = \kappa(D) \). If \( D \) is trivial, then \( \kappa = 1 \) and \( C(D^m) \) is trivial with the sets of imprimitivity of \( D \) as the vertex for any positive integer \( m \). Thus the lemma immediately holds for a trivial digraph. Suppose that \( D \) is not trivial and \( A \) is the adjacency matrix of \( D \). Then by (3), for each positive integer \( s \), \( D^{SM} \) is the disjoint union of digraphs with all possible arcs for \( M \) defined in (2). Thus \( C(D^{SM}) \) is the disjoint union of complete graphs whose vertex sets are \( V(D_1), V(D_2), \ldots, V(D_k) \), respectively. As \( V(D_1), V(D_2), \ldots, V(D_k) \) are the set of imprimitivity of \( D \), we complete the proof. \( \square \)

Now we are ready to prove Theorem 2.3:

**Proof of Theorem 2.3.** By Theorem 2.1, \( \{ C(D^m) \}_{m=1}^{\infty} \) converges. By Lemma 2.6, there exists a positive integer \( M \) such that \( C(D^{SM}) \) has exactly \( \kappa(D) \) components all of which are complete for each positive integer \( s \). Thus a graph with exactly \( \kappa(D) \) components all of which are complete is the limit of a subsequence \( \{ C(D^{SM}) \}_{s=1}^{\infty} \) of \( \{ C(D^m) \}_{m=1}^{\infty} \) and must be the limit of \( \{ C(D^m) \}_{m=1}^{\infty} \). \( \square \)

3. \( \{ \Gamma(A^m) \}_{m=1}^{\infty} \) for a matrix \( A \in B_n \) whose digraph has exactly two strong components

In this section, we study the convergence of \( \{ \Gamma(A^m) \}_{m=1}^{\infty} \) for a matrix \( A \in B_n \) such that

\[
PAP^T = \begin{bmatrix}
A_1 & F \\
0 & A_2
\end{bmatrix}
\]

for a permutation matrix \( P \) of order \( n \), a matrix \( F \), irreducible matrices \( A_1 \) and \( A_2 \). Again, we do so by studying the convergence of \( \{ C(D^m) \}_{m=1}^{\infty} \) for the digraph \( D \) of \( A \) which has exactly two strong components.

If a digraph \( D \) has two strong components and its underlying graph is disconnected, then \( \{ C(D^m) \}_{m=1}^{\infty} \) converges and the limit is the disjoint union of complete graphs as shown in Section 2. Thus, throughout this section, we only consider a weakly connected digraph whose underlying graph is connected.

Throughout this section, for a digraph \( D \) with exactly two strong components, we denote the components by \( D_1 \) and \( D_2 \) so that there is no arc from a vertex in \( D_2 \) to a vertex in \( D_1 \). For \( i = 1, 2 \) and for the positive integer \( M \), defined in (2), \( C(D_i^{SM}) \) has exactly \( \kappa(D_i) \) components all of which are complete for each positive integer \( s \). We denote the sets of imprimitivity of \( D_i \) by \( U_1^{(i)}, U_2^{(i)}, \ldots, U_{\kappa(D_i)}^{(i)} \).

In the rest of this section, we characterize a digraph \( D \) for which \( \{ C(D^m) \}_{m=1}^{\infty} \) converges, and go further to present its limit when \( \{ C(D^m) \}_{m=1}^{\infty} \) converges.

If both \( D_1 \) and \( D_2 \) are trivial, then it is clear that \( C(D^m) \) is an edgeless graph with two vertices for any integer \( m \). That is, if both \( D_1 \) and \( D_2 \) are trivial then \( \{ C(D^m) \}_{m=1}^{\infty} \) converges and the limit graph is an edgeless graph with two vertices. Thus, from now on, we only consider a digraph with exactly two strong components, at least one of which is nontrivial.
We completely characterize a digraph $D$ with two strong components for which $\{C(D^m)\}_{m=1}^\infty$ converges. For a digraph $D$ and a vertex $v$ of $D$, $N_D^-(v)$ denotes the set of all in-neighbors of $v$.

**Lemma 3.1.** Let $D$ be a weakly connected digraph with exactly two strong components $D_1$ and $D_2$. For any two vertices $u \in U_j^{(i)}$ and $v \in U_k^{(i)}$, the length of a directed $(u, v)$-walk is congruent to $k - j$ modulo $\kappa(D_i)$.

**Proof.** Since there is no arc from a vertex in $D_2$ to a vertex in $D_1$, a directed $(u, v)$-walk belongs to $D_i$. Since $D_i$ is strongly connected, we may apply one of the known properties of a strongly connected digraph with the index of imprimitivity $\kappa(D_i)$ to verify the statement. □

**Lemma 3.2.** Let $D$ be a weakly connected digraph with exactly two strong components $D_1$ and $D_2$. Then there exists an integer $M$ such that $C(D^m)$ contains complete graphs whose vertex sets are $U_1^{(1)}$, $U_2^{(1)}$, $U_1^{(2)}$, $U_2^{(2)}$, respectively, as subgraphs for $m \geq M$.

**Proof.** By Theorem 2.3, there exists an integer $M$ such that $C(D^m)$ and $C(D^m')$ equal disjoint union of complete graphs whose vertex sets are $U_1^{(1)}$, $U_2^{(1)}$, and complete graphs whose vertex sets are $U_1^{(2)}$, $U_2^{(2)}$, respectively, for $m \geq M$. Since $C(D^m)$ contains $C(D_1^m)$ and $C(D_2^m)$ as subgraphs for any positive integer $m$, the lemma holds. □

We also need the following lemma:

**Lemma 3.3** [2, Lemma 3.4.3]. Let $D$ be a nontrivial strongly connected digraph, and $U_1$, $U_2$, $U_k(D)$ be the sets of imprimitivity of $D$. Then there exists a positive integer $N$ such that if $x$ and $y$ are vertices belonging respectively to $U_i$ and $U_j$, then there are directed $(x, y)$-walks of every length $j - i + \kappa(D)$ with $t \geq N$.

**Proposition 3.4.** Let $D$ be a weakly connected digraph with exactly two strong components $D_1$ and $D_2$ where $D_1$ is nontrivial and $D_2$ is trivial. Then $\{C(D^m)\}_{m=1}^\infty$ converges if and only if for the vertex $v$ of $D_2$, $|\{i | U_i^{(1)} \cap N_D^-(v) \neq \emptyset\}| = 1$ or $\kappa(D_1)$. Moreover, the limit of $\{C(D^m)\}_{m=1}^\infty$ is a graph consisting of complete components if it converges.

**Proof.** Since $D_1$ is nontrivial, by Lemma 3.3, there exists a positive integer $N$ for which the following holds:

If $x$ and $y$ are vertices belonging respectively to $U_i^{(1)}$ and $U_j^{(1)}$, then there are directed $(x, y)$-walks of every length $j - i + \kappa(D)$ with $t \geq N$.

We show the ‘if’ part of the proposition statement. By Lemma 3.2, there exists an integer $M$ such that $C(D^m)$ contains the complete subgraphs whose vertex sets $U_1^{(1)}$, $U_2^{(1)}$, $U_1^{(2)}$, respectively, as subgraphs for $m \geq M$. In addition, there is no edge joining a vertex in $D_1$ and $v$ in $C(D^m)$ for any positive integer $m$ since $v$ has no out-neighbor.

Suppose that $|\{i | U_i^{(1)} \cap N_D^-(v) \neq \emptyset\}| = 1$. Let $\{i | U_i^{(1)} \cap N_D^-(v) \neq \emptyset\} = \{i^*\}$. Then for any directed walk from a vertex in $U_i^{(1)}$ to $v$, the term right before $v$ on the sequence belongs to $U_{i^*}^{(1)}$ and so it has length congruent to $i^* - i + 1$ modulo $\kappa(D_1)$. Since $i^* - i + 1 \not \equiv i^* - j + 1 \pmod {\kappa(D_1)}$ if $i \not \equiv j \pmod {\kappa(D_1)}$, two vertices belonging to distinct sets of imprimitivity cannot have an $m$-step common prey for any positive integer $m$ and so there is no edge joining two vertices in distinct sets of imprimitivity in $C(D^m)$ for any positive integer $m$. Thus $\{C(D^m)\}_{m=1}^\infty$ converges to the disjoint union of complete graphs whose vertex sets are $U_1^{(1)}$, $U_2^{(1)}$, $\{v\}$. 

**Proof.** Since $D_1$ is nontrivial, by Lemma 3.3, there exists a positive integer $N$ for which the following holds:

If $x$ and $y$ are vertices belonging respectively to $U_i^{(1)}$ and $U_j^{(1)}$, then there are directed $(x, y)$-walks of every length $j - i + \kappa(D)$ with $t \geq N$.

We show the ‘if’ part of the proposition statement. By Lemma 3.2, there exists an integer $M$ such that $C(D^m)$ contains the complete subgraphs whose vertex sets $U_1^{(1)}$, $U_2^{(1)}$, $U_1^{(2)}$, respectively, as subgraphs for $m \geq M$. In addition, there is no edge joining a vertex in $D_1$ and $v$ in $C(D^m)$ for any positive integer $m$ since $v$ has no out-neighbor.

Suppose that $|\{i | U_i^{(1)} \cap N_D^-(v) \neq \emptyset\}| = 1$. Let $\{i | U_i^{(1)} \cap N_D^-(v) \neq \emptyset\} = \{i^*\}$. Then for any directed walk from a vertex in $U_i^{(1)}$ to $v$, the term right before $v$ on the sequence belongs to $U_{i^*}^{(1)}$ and so it has length congruent to $i^* - i + 1$ modulo $\kappa(D_1)$. Since $i^* - i + 1 \not \equiv i^* - j + 1 \pmod {\kappa(D_1)}$ if $i \not \equiv j \pmod {\kappa(D_1)}$, two vertices belonging to distinct sets of imprimitivity cannot have an $m$-step common prey for any positive integer $m$ and so there is no edge joining two vertices in distinct sets of imprimitivity in $C(D^m)$ for any positive integer $m$. Thus $\{C(D^m)\}_{m=1}^\infty$ converges to the disjoint union of complete graphs whose vertex sets are $U_1^{(1)}$, $U_2^{(1)}$, $\{v\}$. 

**Proof.** Since $D_1$ is nontrivial, by Lemma 3.3, there exists a positive integer $N$ for which the following holds:

If $x$ and $y$ are vertices belonging respectively to $U_i^{(1)}$ and $U_j^{(1)}$, then there are directed $(x, y)$-walks of every length $j - i + \kappa(D)$ with $t \geq N$.

We show the ‘if’ part of the proposition statement. By Lemma 3.2, there exists an integer $M$ such that $C(D^m)$ contains the complete subgraphs whose vertex sets $U_1^{(1)}$, $U_2^{(1)}$, $U_1^{(2)}$, respectively, as subgraphs for $m \geq M$. In addition, there is no edge joining a vertex in $D_1$ and $v$ in $C(D^m)$ for any positive integer $m$ since $v$ has no out-neighbor.

Suppose that $|\{i | U_i^{(1)} \cap N_D^-(v) \neq \emptyset\}| = 1$. Let $\{i | U_i^{(1)} \cap N_D^-(v) \neq \emptyset\} = \{i^*\}$. Then for any directed walk from a vertex in $U_i^{(1)}$ to $v$, the term right before $v$ on the sequence belongs to $U_{i^*}^{(1)}$ and so it has length congruent to $i^* - i + 1$ modulo $\kappa(D_1)$. Since $i^* - i + 1 \not \equiv i^* - j + 1 \pmod {\kappa(D_1)}$ if $i \not \equiv j \pmod {\kappa(D_1)}$, two vertices belonging to distinct sets of imprimitivity cannot have an $m$-step common prey for any positive integer $m$ and so there is no edge joining two vertices in distinct sets of imprimitivity in $C(D^m)$ for any positive integer $m$. Thus $\{C(D^m)\}_{m=1}^\infty$ converges to the disjoint union of complete graphs whose vertex sets are $U_1^{(1)}$, $U_2^{(1)}$, $\{v\}$. 

**Proof.** Since $D_1$ is nontrivial, by Lemma 3.3, there exists a positive integer $N$ for which the following holds:

If $x$ and $y$ are vertices belonging respectively to $U_i^{(1)}$ and $U_j^{(1)}$, then there are directed $(x, y)$-walks of every length $j - i + \kappa(D)$ with $t \geq N$.
Now suppose that \(|\{i \mid U_1^{(1)}(i) \cap N_D(v) \neq \emptyset\}| = \kappa(D_1)|. Let u_1, \ldots, u_{\kappa(D_1)} be in-neighbors of \(v\) in \(U_1^{(1)}\), ..., \(U_{\kappa(D_1)}^{(1)}\), respectively. Take a vertex \(x\) of \(D_1\). Then \(x \in U_1^{(1)}\) for some \(j \in \{1, 2, \ldots, \kappa(D_1)\}\) and, by (*) there are directed \((x, u_k)\)-walks of every length \(k - j + t \kappa(D_1)\) with \(t \geq N\) for each \(k = 1, \ldots, \kappa(D_1)\). Thus \(v\) is an \(m\)-step prey of \(x\) for each \(m \geq N\kappa(D_1) + 1\). Since \(x\) is arbitrarily chosen, \(C(D^m)\) is the union of two complete graphs whose vertex sets are \(V(D_1)\) and \([v]\) for each \(m \geq N\kappa(D_1) + 1\) and so \(K_{|V(D_1)|} \cup [v]\) is the limit of \(\{C(D^m)\}_m=1^\infty\).

We show the ‘only if’ part. Suppose that \(|\{i \mid U_1^{(1)}(i) \cap N_D(v) \neq \emptyset\}| = t\) for some \(t \in \{2, \ldots, \kappa(D_1) - 1\}\). Then there exists \(j\) such that \(v\) has in-neighbors in \(U_j^{(1)}\) and \(U_{j+r}^{(1)}\) but no in-neighbor in \(U_{j+1}^{(1)}\) where \(r \in \{2, 3, \ldots, \kappa(D_1) - 1\}\). Take a vertex \(u\) in \(U_j^{(1)}\) and a vertex \(w\) in \(U_{j+r}^{(1)}\). Then, by (*), \(v\) is a \((\kappa(D_1) + 1)\)-step common prey of \(u\) and \(w\) for each \(t \geq N\). Thus \(u\) and \(w\) are adjacent in \(C(D^t\kappa(D_1)+1)\) for each \(t \geq N\). However, \(u\) and \(w\) are not adjacent in \(C(D^{\kappa(D_1)+2})\) for any positive integer \(t \geq N\). To show it, note that \(v\) is the only possible \(m\)-step common prey of \(u\) and \(w\) for any integer \(m\) and so suppose that \(v\) is a \((\kappa(D_1) + 2)\)-step prey of \(u\) for some positive integer \(t \geq N\). We will reach a contradiction. By our assumption, there is a directed \((u, v)\)-walk of length \(\kappa(D_1) + 2\) in \(D\). The vertex immediately followed by \(v\) on the walk must belong to \(U_{j+1}^{(1)}\), which contradicts our assumption that \(v\) does not have an in-neighbor in \(U_{j+1}^{(1)}\). Thus \(v\) cannot be a \((\kappa(D_1) + 2)\)-step prey of \(u\) for any positive integer \(t \geq N\). Therefore \(u\) and \(w\) are not adjacent in \(C(D^{\kappa(D_1)+2})\) for any positive integer \(t \geq N\). Hence we can conclude that \(\{C(D^m)\}_m=1^\infty\) does not converge. \(\square\)

If \(D_2\) is nontrivial, then \(\{C(D^m)\}_m=1^\infty\) converges by Theorem 2.1. Thus we have completely characterized a digraph \(D\) with exactly two strong components for which \(\{C(D^m)\}_m=1^\infty\) converges.

**Theorem 3.5.** Let \(D\) be a weakly connected digraph with exactly two strong components \(D_1\) and \(D_2\) and without arc from \(D_2\) to \(D_1\). Then \(\{C(D^m)\}_m=1^\infty\) converges if and only if \(D\) satisfies one of the following:

(i) \(D_2\) is nontrivial.

(ii) \(D_2\) is trivial and, for the vertex \(v\) of \(D_2\), \(|\{i \mid U_1^{(1)}(i) \cap N_D(v) \neq \emptyset\}| = 1\) or \(\kappa(D_1)\).

From the proof of Proposition 3.4 and Theorem 3.5, we obtain the following:

**Corollary 3.6.** Let \(A\) be a Boolean \((0, 1)\)-matrix in the following form:

\[
\begin{bmatrix}
A_1 & F \\
O & A_2
\end{bmatrix}
\]

where \(O\) is a zero matrix, \(F\) is a nonzero matrix, and \(A_1\) and \(A_2\) are irreducible. Then \(\{\Gamma(A^m)\}_m=1^\infty\) converges if and only if one of the following holds:

(i) \(A_2\) has order at least 2.

(ii) \(A_2\) has order 1 and there exists a permutation matrix \(P\) such that

\[
PAP^T = \begin{bmatrix}
A_{12} & 0 & 0 & \cdots & 0 & 0 \\
0 & A_{23} & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & A_{\kappa(D_1)-2\kappa(D_1)-1} & * \\
* & 0 & 0 & \cdots & 0 & A_{\kappa(D_1)-1\kappa(D_1)} \\
0 & 0 & 0 & \cdots & 0 & A_2
\end{bmatrix}
\]
Lemma 3.7. Let $G$ be an expansion of some bipartite graph $B$ for some vertices $u$ are edges of $B$. Proof. \( z \) vertices $w$ of matrix give in (4) where $D$ each vertex of $x$, $y$, $z$ is nontrivial. Let which is obtained by substituting $I$ if $D$ to make clear that $k$ is trivial, then $1$ is nontrivial, then or each of $A_{12}, A_{23}, A_{k(D_1) - 1, k(D_1)}$. Moreover, the limit of $\{\Gamma(A^m)\}^\infty_{m=1}$ is a JBD matrix.

We now examine the structure of $\{\Gamma(A^m)\}^\infty_{m=1}$ when it converges where $A$ is a Boolean $(0, 1)$-matrix given in (4) where $F$ is a nonzero matrix. If $A_2$ is a trivial matrix, then we presented the limit of $\{\Gamma(A^m)\}^\infty_{m=1}$ in Corollary 3.6. Thus in the following, we find the limit when $A_2$ is nontrivial. To do so, we find the limit of $\{C(D^m)\}^\infty_{m=1}$ where $D$ is the digraph of $A$ and need the following useful notions.

Given a bipartite $B = (X, Y)$, we construct a supergraph of $B$ as follows. We write each edge of $B$ in the arc form $(x, y)$ to make clear that $x \in X$ and $y \in Y$. Then we replace each vertex $v$ with a complete graph $G_v$ (of any size) so that $G_v$, and $G_w$ are vertex-disjoint if $v \neq w$, and join each vertex of $G_v$ and each vertex of $G_w$ whenever either $(x, y)$ is an edge of $B$ or there exists $z \in Y$ such that $(x, z)$ and $(y, z)$ are edges of $B$. We say that the resulting graph $D$ is an expansion of $B$. (See Fig. 2 for an illustration.)

**Lemma 3.7.** Let $G$ be an expansion of some bipartite graph $B = (X, Y)$. Then $G$ has only complete components if and only if for each vertex $x \in X$, the degree of $x$ is at most one in $B$.

**Proof.** We show the ‘if’ part by contradiction. Suppose that there exist vertices $x$, $y$, $z$ such that $xy$ and $xz$ are edges of $G$ but $y$ is not adjacent to $z$ in $G$. Let $G_u$, $G_y$, and $G_w$ be the complete graphs replacing vertices $u$, $v$, and $w$ of $B$ containing $x$, $y$, $z$, respectively. By definition, $u$, $v$, and $w$ are distinct. Since $y$ and $z$ are not adjacent while $x$ is adjacent to both $y$ and $z$, it is true that $u \in X$. Then, since $B$ is bipartite, $v$ and $w$ belong to $Y$. Now, by definition, $u$ is adjacent to $v$ and $w$ and we reach a contradiction.

To show the ‘only if’ part, suppose that $G$ has only complete components and there exists a vertex $u \in X$ which has two neighbors $v$, $w$ in $Y$. Then, by definition, no vertex of $G_v$ is joined to any vertex of $G_w$. Take a vertex $x \in G_u$, a vertex $y \in G_v$, and a vertex $z \in G_w$. Then, by definition, $x$ is adjacent to $y$ and $z$ in $G$ and so $x$, $y$, $z$ belong to the same component. Since $G$ has only complete components by our assumption, $y$ and $z$ are adjacent in $G$, a contradiction. \( \square \)

**Definition 3.8.** We take a weakly connected digraph $D$ with exactly two strong components $D_1$ and $D_2$ where $D_2$ is nontrivial. Let $I(D) = \{(k, l) \mid (x, y) \in A(D) \text{ for some } x \in U_k^{(1)}, y \in U_l^{(2)}\}$. Let $B_D = (Z_{\kappa(D_1)}, Z_{\kappa(D_2)})$ be the bipartite graph defined as follows. If $D_1$ is nontrivial, then $B_D$ has an edge $(i, j)$ if and only if $i \equiv k + 1 + p \pmod{\kappa(D_1)}$ and $j \equiv l + p \pmod{\kappa(D_2)}$ for some $(k, l) \in I(D)$ and some integer $p$. If $D_1$ is trivial, then $B_D$ has an edge $(i, j)$ if and only if $j \equiv l - 1 \pmod{\kappa(D_2)}$ for some $(1, l) \in I(D)$, which is obtained by substituting $p = -1$ and $k(D_1) = 1$ in the nontrivial case.

We note that when we consider an edge $(x, y)$ of $B_D$, the first component and the second component are reduced modulo $\kappa(D_1)$ and $\kappa(D_2)$, respectively. Then following is true.

**Lemma 3.9.** Let $D$ be a weakly connected digraph with exactly two strong components $D_1$ and $D_2$ where $D_2$ is nontrivial. Then $(i, j)$ is an edge of $B_D$ if and only if there exists a $(u, v)$-walk of length $2\kappa(D_1)\kappa(D_2)$ for some vertices $u \in U_i^{(1)}$ and $v \in U_j^{(2)}$ and for some positive integer $s$. 

**Fig. 2.** A bipartite graph $B$ and an expansion $G$ of $B$. 

So that the rows containing nonzero elements of $F'$ intersect exactly one of $A_{12}, A_{23}, \ldots, A_{\kappa(D_1) - 1, \kappa(D_1)}$ or each of $A_{12}, A_{23}, \ldots, A_{\kappa(D_1) - 1, \kappa(D_1)}$. Moreover, the limit of $\{\Gamma(A^m)\}^\infty_{m=1}$ is a JBD matrix.
Proof. For simplicity, we denote $\kappa(D_1)\kappa(D_2)$ by $\lambda$. To show the 'only if' part, suppose that $(i, j)$ is an edge of $B_D$. By definition, for some integer $p$ and for some $(k, l) \in I(D)$,

$$i \equiv k + 1 + p \pmod{\kappa(D_1)}, \quad j \equiv l + p \pmod{\kappa(D_2)}.$$ 

We define nonnegative integers $p_1$ and $p_2$ as follows. There exists a positive integer $s$ such that both $s\lambda - p - 1$ and $s\lambda + p$ are positive integers. If $D_1$ is trivial, then let $p_1 = 0$ and $p_2 = 2s\lambda + p$ (note that $p = -1$). If $D_1$ is nontrivial, then let $p_1 = s\lambda - p - 1$ and $p_2 = s\lambda + p$.

Since $(k, l) \in I(D)$, there exist vertices $x \in U_k^{(1)}$ and $y \in U_l^{(2)}$ such that $(x, y)$ is an arc of $D$. If $D_1$ is trivial, then let $u = x$ and then $Q := u$ is a $(u, x)$-walk of length $p_1 = 0$. Suppose that $D_1$ is nontrivial. Since $D_1$ is a nontrivial strongly connected digraph, any vertex in $D_1$ has an in-neighbor in $D_1$ and so there is a directed $(u, x)$-walk $Q$ of length $p_1$ for some $u \in V(D_1)$. Since $x \in U_k^{(1)}$, it is true that $u \in U_k^{(1)}$. However,

$$k - p_1 \equiv (i - p - 1) - (s\lambda - p - 1) \equiv i \pmod{\kappa(D_1)}.$$ 

Therefore $u \in U_k^{(1)}$.

Since $D_2$ is nontrivial and strongly connected $D_2$ has a directed $(y, v)$-walk $R$ of length $p_2$ for some $v \in V(D_2)$. Therefore $y \in U_l^{(2)}$. However,

$$l + p_2 \equiv (j - p) + (ks\lambda + p) \equiv j \pmod{\kappa(D_2)},$$

where $k = 1$ if $D_1$ is nontrivial and $k = 2$ if $D_1$ is trivial. Thus $v \in U_l^{(2)}$.

To show the 'if' part, suppose that there exists a directed $(u, v)$-walk $Q$ of length $2s\lambda$ for some vertices $u \in U_i^{(1)}$ and $v \in U_j^{(2)}$ and for some positive integer $s$. The walk $Q$ contains a unique arc $(w, z)$ such that $w \in V(D_1)$ and $z \in V(D_2)$ since there is no arc from a vertex in $D_2$ to a vertex in $D_1$. Then $w \in U_r^{(1)}$ and $z \in U_s^{(2)}$ for some $r \in \{1, 2, \ldots, \kappa(D_1)\}$ and some $s \in \{1, 2, \ldots, \kappa(D_2)\}$. By definition, $(r, s) \in I(D)$. Let $\ell_1$ and $\ell_2$ be the lengths of $(u, w)$-section of $Q$ and $(z, v)$-section of $Q$, respectively. Then

$$i + \ell_1 \equiv r \pmod{\kappa(D_1)}, \quad j - \ell_2 \equiv s \pmod{\kappa(D_2)}.$$ 

Since $\ell_1 + \ell_2 + 1 = 2s\lambda$, the first congruence relation is equivalent to $i + (2s\lambda - \ell_2 - 1) \equiv r \pmod{\kappa(D_1)}$ and so $i - \ell_2 - 1 \equiv r \pmod{\kappa(D_1)}$. Therefore

$$i \equiv r + \ell_2 + 1 \pmod{\kappa(D_1)}, \quad j \equiv s + \ell_2 \pmod{\kappa(D_2)}.$$ 

By definition, $(i, j)$ is an edge of $B_D$. □

Now we are ready to present the limit of $\{\Gamma(A^m)\}_{m=1}^{\infty}$ if it exists for a matrix $A$ given in (4) when $F$ is a nonzero matrix and $A_2$ has order at least 2, that is, $D(A_2)$ is a nontrivial strong component of $D(A)$:

**Theorem 3.10.** Let $D$ be a weakly connected digraph with two strong components $D_1$ and $D_2$ such that no arc goes from $D_2$ to $D_1$, $D_2$ is nontrivial, and $|C(D^m)|_{m=1}^{\infty}$ converges. Then the limit of $|C(D^m)|_{m=1}^{\infty}$ is an expansion of the bipartite graph $B_D$ defined in Definition 3.8.

**Proof.** Let $G$ be the limit of $|C(D^m)|_{m=1}^{\infty}$. By Theorem 3.2, complete graphs whose vertex sets are $U_k^{(1)}$, $U_k^{(2)}$, respectively, are subgraphs of $G$. 


Since there is no arc from a vertex in $D_2$ to a vertex in $D_1$, for any positive integer $m$, an $m$-step common prey of two vertices of $V(D_2)$ is in $D_2$ and so the union of complete graphs whose vertex sets are $U_1^{(2)}, \ldots, U_{\kappa(D_2)}^{(2)}$, respectively, is an induced subgraph of $G$ by Theorem 2.3. Therefore, to show that $G$ is an expansion of $B_D$, it is sufficient to prove the following:

(i) $x \in U_i^{(1)}$ and $y \in U_j^{(2)}$ are adjacent in $G$ if and only if $(i, j)$ is an edge of $B_D$.

(ii) For distinct $i$ and $j$, $x \in U_i^{(1)}$ and $y \in U_j^{(1)}$ are adjacent in $G$ if and only if $(i, h)$ and $(j, h)$ are edges of $B_D$ for some $h \in \mathbb{Z}_{\kappa(D_2)}$.

For simplicity, we denote $\kappa(D_1)\kappa(D_2)$ by $\lambda$.

To show (i), suppose that $x \in U_i^{(1)}$ and $y \in U_j^{(2)}$ are adjacent in $G$. Then there exist an integer $s$ and a $2s\lambda$-step common prey $z$ of $x$ and $y$ in $D$, which implies that there exists an $(x, z)$-walk $Q$ of length $2s\lambda$ in $D$. On the other hand, since $z$ is a $2s\lambda$-step prey of $y$, it is true that $z \in V(D_2)$. Furthermore, since $2s\lambda$ is a multiple of $\lambda$, $z \in U_i^{(2)}$. By Lemma 3.9, $(i, j)$ is an edge of $B_D$.

Suppose that $(i, j)$ is an edge of $B_D$. Take vertices $x \in U_i^{(1)}$ and $y \in U_j^{(2)}$. By Lemma 3.9, there exists a $(u, v)$-walk $Q$ of length $2s\lambda$ for some vertices $u \in U_i^{(1)}$ and $v \in U_j^{(2)}$ and some positive integer $s$. Since any directed $(x, u)$-walk belongs to $D_1$, the length of a directed $(x, u)$-walk is congruent to 0 modulo $\kappa(D_1)$. Then there exists a directed $(x, u)$-walk $S$ of length $s'\lambda$ for some integer $s'$ (if $D_1$ is trivial then $s' = 0$). Since $D_2$ is nontrivial and $\kappa(D_2)$ divides $\lambda$, by Lemma 3.3, for some integer $N$, there exist a directed $(v, y)$-walk $T$ of length $N\lambda$, and a directed $(v, y)$-walk $R$ of length $(2s + s' + N)\lambda$. Since $\text{SQT}$ is a directed $(v, y)$-walk of length $(2s + s' + N)\lambda$, $\nu$ is a $(2s + s' + N)\lambda$-step common prey of $x$ and $y$ in $D$, and so $x$ and $y$ are adjacent in $G$. Hence (i) holds.

Now we show that (ii) holds. Take distinct $i$ and $j$ in $\{1, \ldots, \kappa(D_1)\}$. To prove the ‘if’ part, suppose that $(i, h)$ and $(j, h)$ are edges of $B_D$ for some $h \in \mathbb{Z}_{\kappa(D_2)}$. By Lemma 3.9, there exist a directed $(u_1, v_1)$-walk $S_1$ of length $2s_1\lambda$ and a directed $(u_2, v_2)$-walk $S_2$ of length $2s_2\lambda$ for some positive integers $s_1$ and $s_2$ and some vertices $u_1 \in U_i^{(1)}$, $u_2 \in U_j^{(1)}$, $v_1, v_2 \in U_h^{(2)}$. Take two vertices $x \in U_i^{(1)}$ and $y \in U_j^{(1)}$. Note that assumption $i \neq j$ implies that $D_1$ is nontrivial. Therefore both $D_1$ and $D_2$ are nontrivial, we may apply Lemma 3.3 to $D_1$ and $D_2$, respectively, to have integers $N_1$ and $N_2$ satisfying the following. Since $v_1, v_2$ belong to the same set of imprimitivity, there exist a directed $(v_1, v_2)$-walk $T_1$ of length $(2s_1 + N_2)\lambda$ and a directed $(v_2, v_3)$-walk $T_2$ of length $(2s_1 + N_2)\lambda$. Then $S_1T_1$ is a directed $(u_1, v_2)$-walk of length $(2s_2 + 2s_1 + N_2)\lambda$ and $S_2T_2$ is a directed $(u_2, v_3)$-walk of length $(2s_2 + 2s_1 + N_2)\lambda$. Take any integer $t \geq N_1$. Then, since $x, u_1$ belong to the same set of imprimitivity, there exists a directed $(x, u_1)$-walk $Q_1$ of length $t\lambda$. For the same reason, there exists a directed $(y, u_2)$-walk $Q_2$ of length $t\lambda$. Now $Q_1S_1T_1$ is a directed $(x, v_2)$-walk of length $(2s_1 + 2s_2 + N_1 + t)\lambda$ and $Q_2S_2T_2$ is a directed $(y, v_3)$-walk of length $(2s_1 + 2s_2 + N_2 + t)\lambda$. Thus $v_2$ is a $(2s_1 + 2s_2 + N_2 + t)\lambda$-step common prey of $x$ and $y$, and hence $x$ and $y$ are adjacent in $G$.

To show the ‘only if’ part, suppose that $x \in U_i^{(1)}$ and $y \in U_j^{(1)}$ are adjacent in $G$. Then they have a $2s\lambda$-step common prey $z$ in $V(D_2)$ for some integer $s$, that is, there exist a directed $(x, z)$-walk and a directed $(y, z)$-walk of length $2s\lambda$. Since $z \in V(D_2)$, it holds that $z \in U_i^{(2)}$ for some $h \in \{1, 2, \ldots, \kappa(D_2)\}$. By Lemma 3.9, $(i, h)$ and $(j, h)$ are edges of $B_D$. □

We can easily check that Theorem 3.10 is equivalent to the following:

**Corollary 3.11.** Let $A \in B_n$ be a matrix such that for a permutation matrix $P$ of order $n$,

$$P \text{AP}^T = \begin{bmatrix} A_1 & F \\ O & A_2 \end{bmatrix}$$
where $O$ is a zero matrix,

$$
A_1 = \begin{bmatrix}
O & A_{12} & 0 & \cdots & 0 \\
O & 0 & A_{23} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
O & 0 & O & \cdots & A_{k(D_1)-1,k(D_1)} \\
A_{k(D_1)1} & 0 & 0 & \cdots & 0
\end{bmatrix},
A_2 = \begin{bmatrix}
0 & B_{12} & 0 & \cdots & 0 \\
0 & 0 & B_{23} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & B_{k(D_2)-1,k(D_2)} \\
B_{k(D_2)1} & 0 & 0 & \cdots & 0
\end{bmatrix},
$$

and $A_2$ has order at least two, and $F$ is a nonzero matrix,

$$
F = \begin{bmatrix}
F_{11} & F_{12} & F_{13} & \cdots & F_{1k(D_2)} \\
F_{21} & F_{22} & F_{23} & \cdots & F_{2k(D_2)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
F_{k(D_1)-1,1} & F_{k(D_1)-1,2} & F_{k(D_1)-1,3} & \cdots & F_{k(D_1)-1,k(D_2)} \\
F_{k(D_1)1} & F_{k(D_1)2} & F_{k(D_1)3} & \cdots & F_{k(D_1)k(D_2)}
\end{bmatrix}.
$$

Then $\{\Gamma(A^m)\}_{m=1}^{\infty}$ converges to a matrix $A'$ such that

$$
PA'P^T = \begin{bmatrix}
C_1 & F' \\
F'^T & C_2
\end{bmatrix}
$$

where

$$
C_1 = \begin{bmatrix}
J & C_{12} & C_{13} & \cdots & C_{1k(D_1)} \\
C_{21} & J & C_{23} & \cdots & C_{2k(D_1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
C_{k(D_1)-1,1} & C_{k(D_1)-1,2} & C_{k(D_1)-1,3} & \cdots & C_{k(D_1)-1,k(D_1)} \\
C_{k(D_1)1} & C_{k(D_1)2} & C_{k(D_1)3} & \cdots & J
\end{bmatrix},
C_2 = \begin{bmatrix}
J & 0 & 0 & \cdots & 0 \\
0 & J & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & J
\end{bmatrix}.
$$

$J$ represents a matrix of an appropriate size with all the elements 1; $C_{ij} = C_{ji} = J$ if $F'_{ik} = F'_{jk} = J$ for some $k \in \{1, \ldots, \kappa(D_2)\}$ and $C_{ij} = C_{ji} = 0$ otherwise; $F'_{ij} = J$ if one of the following holds:

- $A_1$ has order at least two and $F_{k,l} \neq O$ for some integers $k$, $l$ satisfying $i \equiv k + p + 1 \pmod{\kappa(D_2)}$ and $j \equiv l + p \pmod{\kappa(D_2)}$ for some integer $p$,
- $A_1$ has order one and $F_{1l} \neq O$ for some integer $l$ such that $j \equiv l - 1 \pmod{\kappa(D_2)}$.  


and $F_{ij} = 0$ otherwise.

Let $A$ be a matrix given in (4) where $F$ is nonzero and $D$ be the digraph of $A$. If $D_1$ is trivial, $D_2$ is nontrivial and $B_D$ has at least two edges, then any expansion of $B_D$ cannot be the union of complete subgraphs and so the limit of $\{D^{(m)}\}_{m=1}^{\infty}$ is not the union of complete subgraphs by Theorem 3.10, that is, the limit of $\{\Gamma(A^m)\}_{m=1}^{\infty}$ cannot be a JBD matrix. If $D_2$ trivial and $\{D^{(m)}\}_{m=1}^{\infty}$ converges, then the limit of $\{\Gamma(A^m)\}_{m=1}^{\infty}$ is always the union of complete subgraphs by Proposition 3.4, that is, the limit of $\{\Gamma(A^m)\}_{m=1}^{\infty}$ is a JBD matrix. In the following, we characterize a matrix $A$ given in (4) for which the limit graph of $\{\Gamma(A^m)\}_{m=1}^{\infty}$ is a JBD matrix when $F$ is a nonzero matrix and both $A_1$ and $A_2$ are nontrivial, that is, we characterize a digraph $D$ with exactly two strong components both of which are nontrivial and for which the limit graph of $\{C(D^{(m)})\}_{m=1}^{\infty}$ has only complete components.

**Theorem 3.12.** Let $D$ be a weakly connected digraph with exactly two strong components $D_1$ and $D_2$ both of which are nontrivial and without arc from $D_2$ to $D_1$. Suppose that $\{C(D^{(m)})\}_{m=1}^{\infty}$ converges to a graph $G$. Then $G$ is the union of complete subgraphs if and only if $\kappa(D_2)$ divides $\kappa(D_1)$ and $i - j \equiv i' - j'$ (mod $\kappa(D_2)$) for any $(i, j), (i', j') \in I(D)$.

**Proof.** As we have shown in the proof of Theorem 3.10, $G$ is an expansion of the bipartite graph $B_D$ defined in Definition 3.8. For convenience, let $B_D = (X, Y)$. Suppose that $\kappa(D_2)|\kappa(D_1)$ and $j - i \equiv j' - i'$ (mod $\kappa(D_2)$) for any $(i, j), (i', j') \in I(D)$. Take a vertex $a \in X$. If $a$ has no neighbor in $B_D$, then its degree is zero. Suppose that $a$ has a neighbor in $B_D$. Let $(a, b)$ and $(a, c)$ are edges of $B_D$. Then by definition, for some $(i, j), (i', j') \in I(D)$, for some integers $l, l'$,

\[
a \equiv i + l + 1 \pmod{\kappa(D_1)}, \quad b \equiv j + l \pmod{\kappa(D_2)},
\]

\[
a \equiv i' + l' + 1 \pmod{\kappa(D_1)}, \quad c \equiv j' + l' \pmod{\kappa(D_2)}.
\]

Since $\kappa(D_2)|\kappa(D_1)$,

\[
a \equiv i + l + 1 \equiv i' + l' + 1 \pmod{\kappa(D_2)},
\]

and so $l - l' \equiv i' - i \pmod{\kappa(D_2)}$. Therefore

\[
b - c \equiv (j + l) - (j' + l') \equiv (j - j') + (l - l') \equiv (j - j') + (i' - i) \equiv 0 \pmod{\kappa(D_2)}.
\]

Therefore, the vertex $a$ has only one neighbor in $B_D$. Hence, by Lemma 3.7, $G$ is the union of complete subgraphs.

Now suppose that $G$ is the union of complete subgraphs. Then, by Lemma 3.7, the degree of each vertex in $X$ is at most one in $B_D$. Since we have assumed that the underlying graph of $D$ is connected at the beginning of this section, $B_D$ has an edge and so there exists a vertex $a \in X$ such that the degree of $a$ is one. Then $(a, b)$ is an edge of $B_D$ and so for some $(i, j) \in I(D)$ and some integer $l$,

\[
a \equiv i + l + 1 \pmod{\kappa(D_1)}, \quad b \equiv j + l \pmod{\kappa(D_2)}.
\]

Since $(i, j) \in I(D)$, it is true that $(i + l + \kappa(D_1) + 1, j + l + \kappa(D_1))$ is an edge of $B_D$ by the definition of $B_D$. Since $i + l + \kappa(D_1) + 1 \equiv a \pmod{\kappa(D_1)}$, it holds that $(a, j + l + \kappa(D_1))$ is an edge of $B_D$. Since the degree of $a$ is one, $b$ is the unique neighbor of $a$ in $B_D$ and so $j + l + \kappa(D_1) \equiv b \pmod{\kappa(D_2)}$. Thus $j + l + \kappa(D_1) \equiv j + l \pmod{\kappa(D_2)}$ and so $\kappa(D_1) \equiv 0 \pmod{\kappa(D_2)}$. Therefore $\kappa(D_2)|\kappa(D_1)$.

Take $(i, j), (i', j') \in I(D)$. Without loss of generality, we may assume that $i' > i$. Since $(i, j) \in I(D)$, by the definition of $B_D$, $(i + (i' - i) + 1, j + (i' - i))$ is an edge of $B_D$ and so is $(i' + 1, j + i' - i)$. In addition, $(i' + 1, j')$ is an edge of $B_D$ as $(i', j') \in I(D)$. Therefore both $(i' + 1, j + i' - i)$ and $(i' + 1, j')$ are edges of $B_D$. Since each vertex in $X$ of $B_D$ has degree at most one, $j + (i' - i) \equiv j' \pmod{\kappa(D_2)}$. Thus $i - j \equiv i' - j' \pmod{\kappa(D_2)}$. \[\square\]

As a corollary of Theorem 3.12, we obtain the following:
Corollary 3.13. Let $A \in B_n$ be a matrix such that for a permutation matrix $P$ of order $n$,

$$PAP^T = \begin{bmatrix} A_1 & F \\ O & A_2 \end{bmatrix}$$

where

$$A_1 = \begin{bmatrix} 0 & A_{12} & 0 & \cdots & 0 \\
0 & 0 & A_{23} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & A_{k(D_1)-1,k(D_1)} \\
A_{k(D_1)1} & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & B_{12} & 0 & \cdots & 0 \\
0 & 0 & B_{23} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & B_{k(D_2)-1,k(D_2)} \\
B_{k(D_2)1} & 0 & 0 & \cdots & 0 \end{bmatrix},$$

both $A_1$ and $A_2$ have order at least two, $O$ is a zero matrix, and $F$ is a nonzero matrix.

Suppose that $\{\Gamma(A^m)\}_{m=1}^{\infty}$ converges to a matrix $A'$. Then $A'$ is a JBD matrix if and only if $\kappa(D_2)$ divides $\kappa(D_1)$ and $i - j \equiv i' - j' \pmod{\kappa(D_2)}$ whenever $F_{i,j} \neq 0$ and $F_{i',j'} \neq 0$.

4. Concluding remarks

In this paper, we investigated the convergence and the limit of the matrix sequence $\{\Gamma(A^m)\}_{m=1}^{\infty}$ for a matrix $A$ in $B_n$ whose digraph $D$ has at most two strong components and, among such matrices, characterized a matrix $A$ for which the limit of $\{\Gamma(A^m)\}_{m=1}^{\infty}$ is a JBD matrix. We would like to see if our results can be generalized for an arbitrary matrix in $B_n$. When a digraph $D$ has quite many strong components, vertices in the strong component which has only outgoing arcs in the condensation of $D$ have much more choices for prey and so the characterization of its limit, if it exists, appears to be more difficult.

We mentioned earlier that studying the matrix sequence $\{\Gamma(A^m)\}_{m=1}^{\infty}$ for a matrix $A$ in $B_n$ is equivalent to studying the graph sequence $\{C(D^m)\}_{m=1}^{\infty}$ and that $\{C(D^m)\}_{m=1}^{\infty}$ is actually the sequence of $m$-step competition graphs of $D$. In this context, we propose to investigate the graph sequence obtained by other variants of competition graph (see [4,9,15]).

Acknowledgments

We wish to acknowledge the anonymous referee for invaluable suggestions leading to improvements in the presentation of the results.

References