MATROIDS AND THE CORRELATION CONSTANT

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Abstract. The correlation constant of a matrix, field, graph, or matroid explains how the edges in a graph or the column vectors in a vector space configuration are correlated. Our goal is to study different matrices and matroids and find correlation constants greater than 1 to improve the lower bound of the correlation constant for any field $F$.

1. Introduction

Let us start with $M$, a finite collection of vectors and $b_i, b_{ij}, b$ stand for the number of bases for the column space of $M$ that contain the distinct vector $i$, the distinct vectors $i$ and $j$, and the total number of bases for the column space of $M$ respectively. The formal definition of a correlation constant of a field $F$ is the supremum of $\frac{bb_{ij}}{b_i b_j}$ over all pairs of distinct vectors $i$ and $j$ in finite vector configurations in vectors spaces over $F$.

The Complete Graphs $K_n$. The first complete graph we will analyze is $K_4$. It can be described by the vertex-edge incidence matrix below.

$$K_4 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix}$$

Each row represents a vertex and each column represents an edge in the complete graph. The entries of this matrix indicated that this will also be a directed graph. If we look at all the entries of the second row we can analyze all the edges related to the second vertex. Edge 1 is directed towards the second vertex because the $(2,1)$-entry is -1. Edge’s 4 and 5 are directed away from the second vertex because the $(2,4)$ and $(2,5)$-entries are 1’s. The $(i,j)$-entries that are 0 indicate that there are no edges incident on that vertex.

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The Rank and The Bases. First we will determine the rank of the matrix in order to find the dimension of the column space. Let us choose to view each column as a column vector and using Gaussian elimination we can see that the rank of the matrix is 3, therefore we will find bases for the column space with 3 vectors.

This graph is represented over all the real numbers so after using Gaussian elimination to determine all the possible bases we can see that each vector is included in 8 bases. There is also a total of 16 bases.

Below, is the Hodge-Riemann form of the vector configuration of $K_4$ from Huh’s paper [1].

$$HR(K_4) = \begin{bmatrix}
0 & 3 & 3 & 3 & 3 & 4 \\
3 & 0 & 3 & 3 & 4 & 3 \\
3 & 3 & 0 & 4 & 3 & 3 \\
3 & 3 & 4 & 0 & 3 & 3 \\
3 & 4 & 3 & 3 & 0 & 3 \\
4 & 3 & 3 & 3 & 3 & 0 
\end{bmatrix}$$

The entries of this matrix is described by

$$HR(M)_{ij} = \begin{cases}
0 & \text{if } i = j, \\
b_{ij} & \text{if } i \neq j.
\end{cases}$$

Now that we have the various combinations of $b_i, b_j$ and $b_{ij}$ we can concoct a set of potential correlation constants and take the supremum to find the correlation constant of $K_4$

$$\sup \left\{0, \frac{3}{4}, 1\right\} = 1$$

2. Patterns of the Correlation Constant

Now we would like to ask ourselves, what essentially affects the correlation constant? Does it depend on the characteristic of a field or is it an intrinsic property of a field? [1]

Modular Arithmetic. Let us see how the correlation constant of $K_4$ will be affected if we were to calculate it in modulo 2. We will call $Z$ the matrix representing $K_4$ over $\mathbb{F}_2$.

$$Z = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 
\end{bmatrix}$$

Now that the (i,j)-entries of $K_4$ are in modulo 2 we can calculate the rank of $Z$ and determine the bases for its column space. The rank of $Z$ is 4 so we must use Gaussian elimination to determine 4 linearly independent vectors that span the column space. However, we must use Gaussian elimination via modulo 2 and from that we can determine that there are no possible bases for the column space. The correlation constant of $K_4$ via modulo 2 is 0. From these observations we can see that the correlation constant of $K_4$ decreased when represented over $\mathbb{F}_2$. 
For the rest of this section we would like to answer the following questions:
(1) Is the correlation constant of every complete graph 1?
(2) Does the correlation constant always decrease with any complete graph when represented over \( \mathbb{F}_2 \)?
(3) Is the correlation constant affected when a complete graph has orientation? Whether it is represented by an incidence matrix or an adjacency matrix?

We will start by answering the first question, and we will look at the incidence matrix of \( K_3 \) which is essentially a triangle.

\[
K_3 = \begin{bmatrix}
1 & 0 & -1 \\
-1 & 1 & 0 \\
0 & -1 & 1
\end{bmatrix}
\]

After repeating the same process as described earlier in this paper we can see that the correlation constant is \( \frac{3}{4} \). Now let us see how the correlation constant is affected when this matrix is represented over \( \mathbb{F}_2 \). Let \( V \) denote the matrix representing \( K_3 \) over \( \mathbb{F}_2 \)

\[
V = \begin{bmatrix}
1 & 0 & 1 \\
1 & 1 & 0 \\
0 & 1 & 1
\end{bmatrix}
\]

We can see that the correlation constant of \( V \) is the same as \( K_3 \), therefore we can answer the second question. The correlation constant does not always decrease when the matrix is represented over \( \mathbb{F}_2 \). Although this appeared to be a promising pattern it is a lot more difficult to find one than it seems. Now let’s see what happens when \( K_4 \) and \( K_3 \) are represented by an adjacency matrix.

Adjacency matrices are square matrices that represent undirected graphs with 0’s and 1’s as its entries. The dimensions of the adjacency matrix representing a complete graph \( K_n \) is \( n \times n \). The following are the adjacency matrices for \( K_4 \) and \( K_3 \).

\[
K_4 = \begin{bmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0
\end{bmatrix}, K_3 = \begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{bmatrix}
\]

Using the same process as before to calculate the bases we can see that the correlation constant of these two matrices are 1. As you calculate the correlation constant of more and more adjacency matrices representing complete graphs we can see that it is always 1. Now that we have seen the incidence and adjacency matrices for \( K_4 \) and \( K_3 \) we can answer questions 1 and 3. The correlation constant of every complete graph \( K_n \) is always 1 when represented by an adjacency matrix (undirected), but only sometimes when represented by an incidence matrix (directed). This leads us to postulate that orientation and direction does affect the correlation constant of a graph.

### 3. Interesting Matroids

Now that we have gained an understanding of what a correlation constant is, how to calculate it, and its properties we can start to look at more complex matroids.
Let us look at the following matroid $S_8$ from Oxley’s book [2].

$$S_8 = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1
\end{bmatrix}$$

This matroid is only representable if and only if that field has characteristic two, therefore it is represented over $GF(2)$. We must now determine the correlation constant of this matroid by using the previous processes, but with Gaussian elimination via modulo 2. We will use Sage to easily calculate the bases of this matroid [4]. From this we can determine a set of potential correlation constants and take the supremum of that set to get the correlation constant which is $\frac{48 \cdot 12}{20 \cdot 28} = \frac{36}{35}$. We obtained these numbers when $i = 4$ and $j = 8$. This is the first correlation constant we have seen greater than 1 so we have accomplished what we wanted to in the abstract.

Our next matroid is called $M_{4,2}$ which can be constructed from the conditions set forth in Schröter’s paper [3]. This was the first matroid discovered by Seymour and Welsh to have a correlation constant greater than 1.

$$M_{4,2} = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1
\end{bmatrix}$$

Using Gaussian elimination via modulo 2 [4] we can see that the correlation constant of this matroid is also $\frac{36}{35}$. We obtained these numbers when $i = 1$ and $j = 5$. $M_{4,2}$ and $S_8$ are represented by two different matrices, so it is curious to see that their correlation constants are the same. However, the first four columns of each matrix were part of the $I_4$ identity matrix which means the other 4 column vectors are linear combinations of the first four.

4. Hypotheses and Conclusions

From what we have seen so far we can see that to find correlation constants greater than 1 we do not have to construct large matrices, nor make the entries of these matrices large. We have seen two examples of matrices completely comprised of only 1’s and 0’s that involve the identity matrix and their correlation constants were greater than 1. We now hope to find more matroids with correlation constants greater than 1 and $\frac{36}{35}$ to improve its lower bound. We have established a solid background, found few patterns, and seen real examples of these special matroids. We hope to find more and have the tools to do so.
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REFERENCES


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