



FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

Tetrises and Graph Coloring

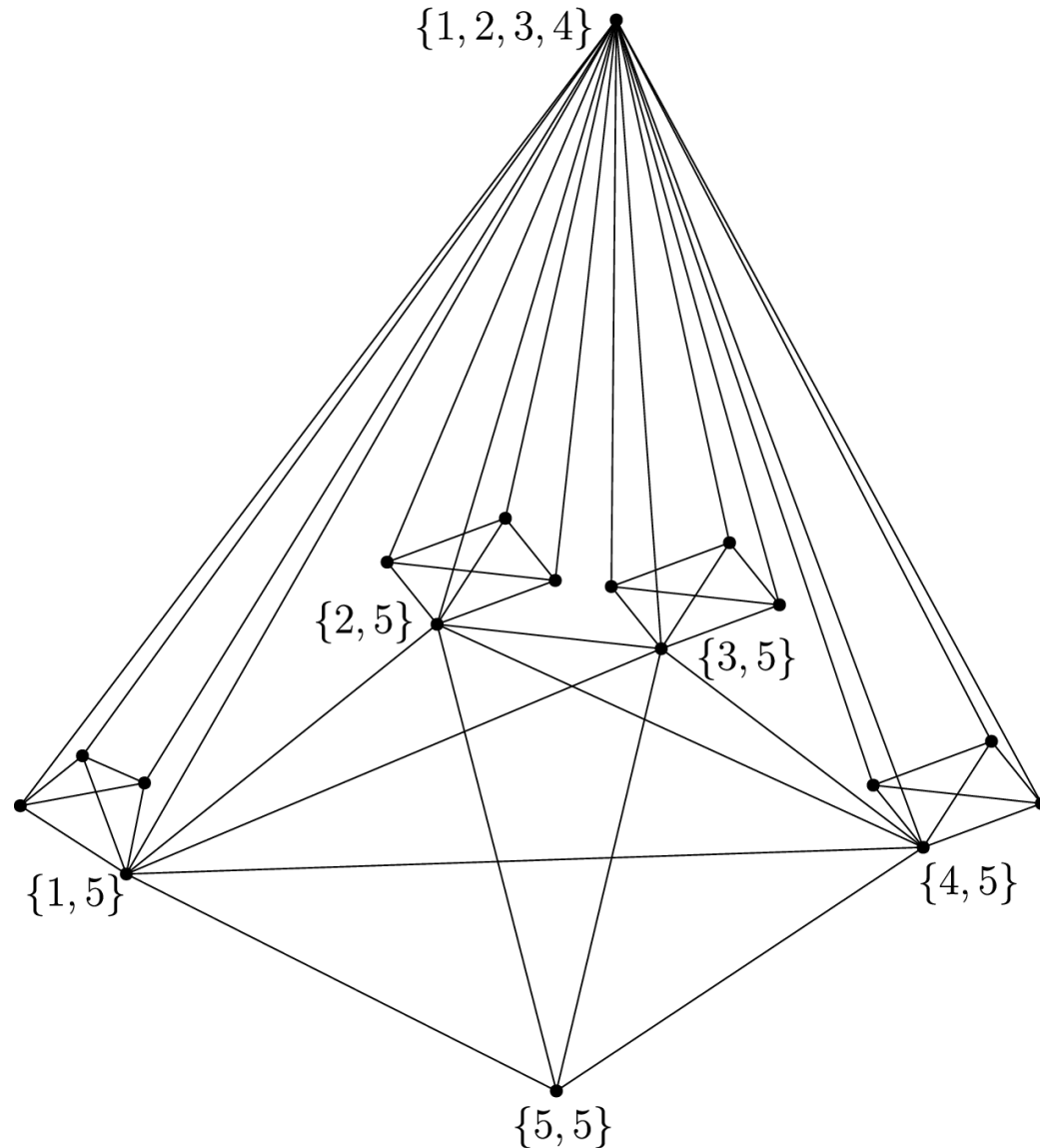
(joke included)

Aneta Štastná, Ondřej Šplíchal

Erdős–Faber–Lovász conjecture

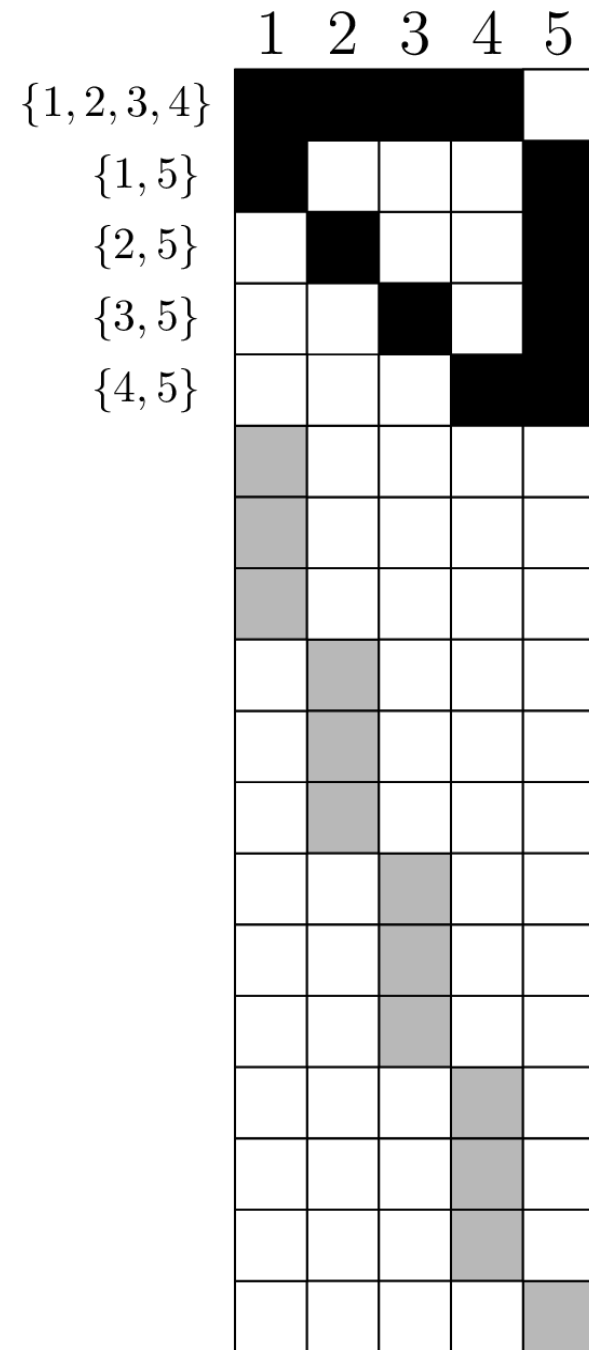
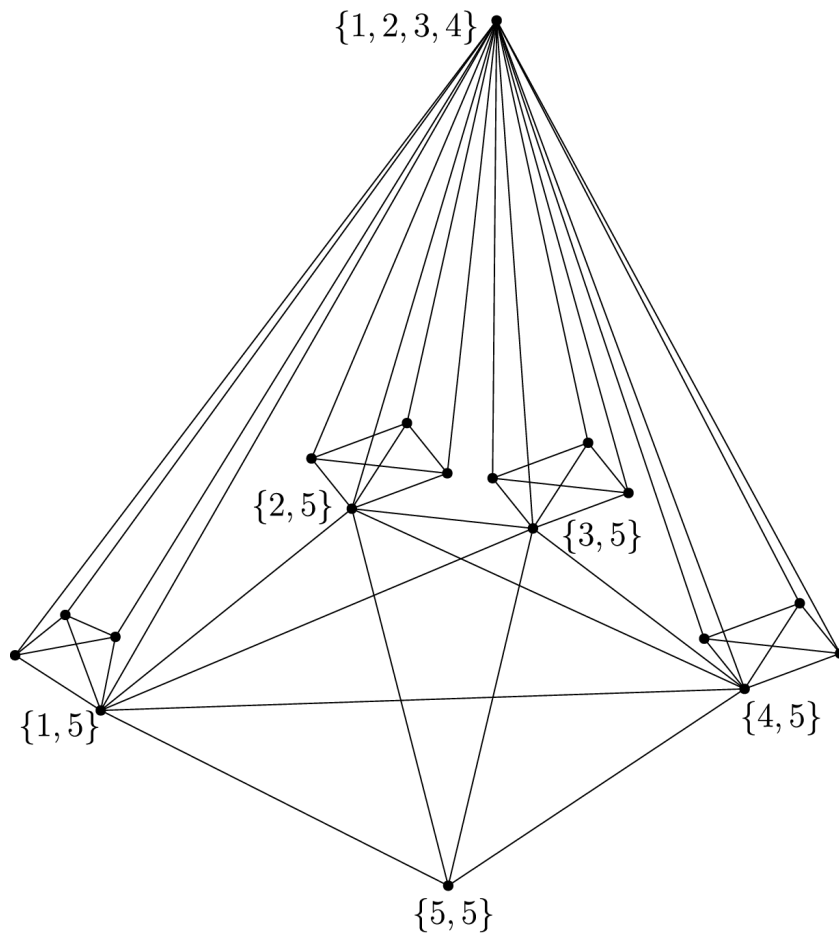
- clique version

- If n complete graphs, each having exactly n vertices, have the property that every pair of complete graphs has at most one shared vertex, then the union of the graphs can be colored with n colors.
- Example: $n = 5$

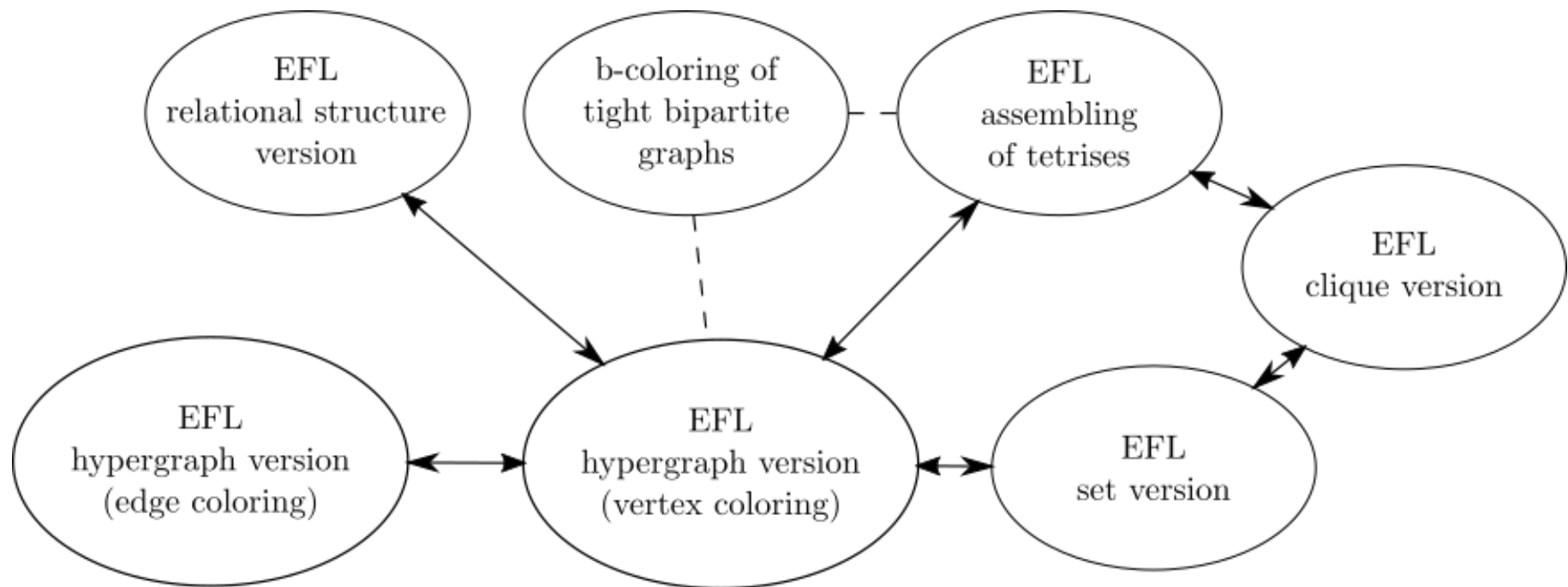


Tetrises

New Perspective on EFL

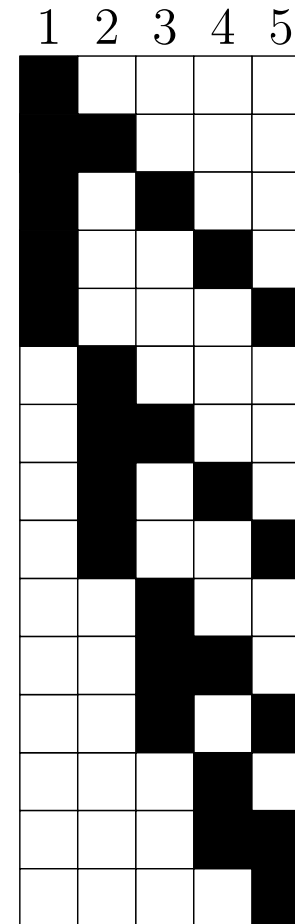


Different perspectives on EFL



Tetris with maximal number of crossings T

- Each two cliques intersect in one vertex.
- No vertex belongs to more than 2 cliques.



Coloring of T

	1	2	3	4	5
1	Black				
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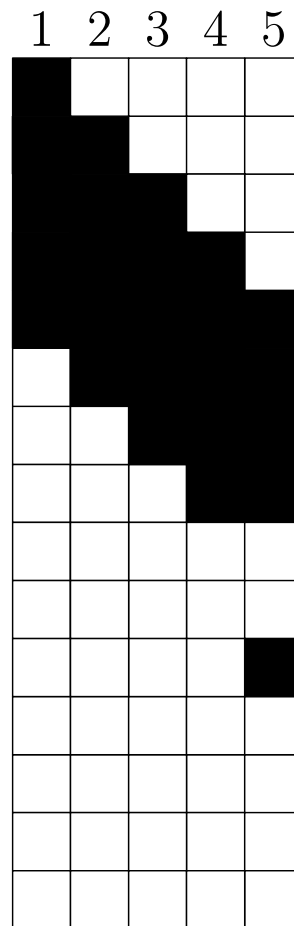
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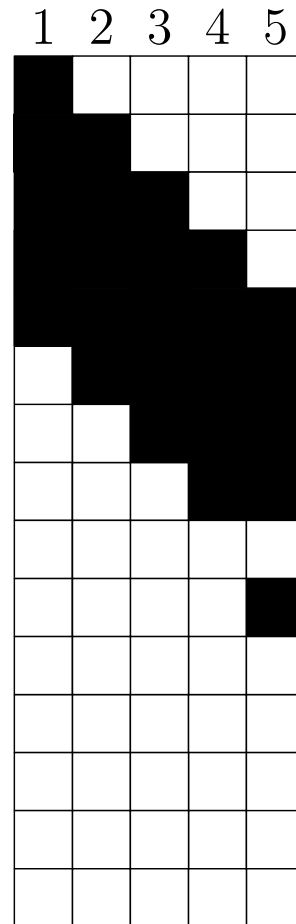
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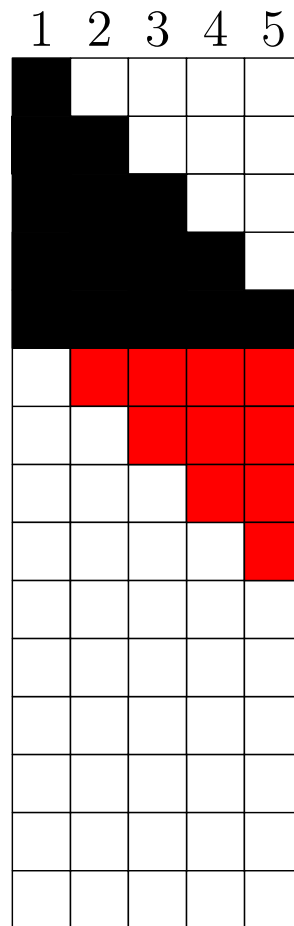
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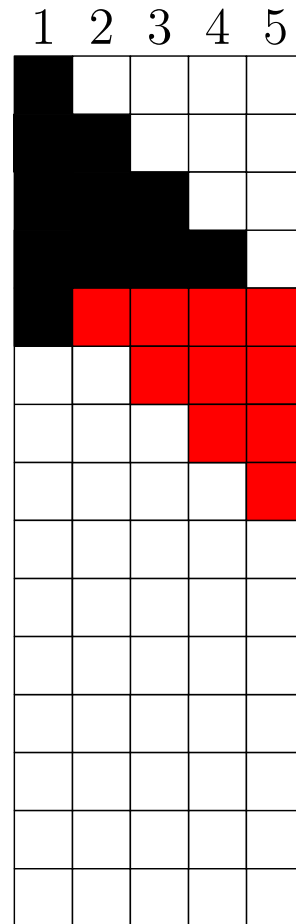
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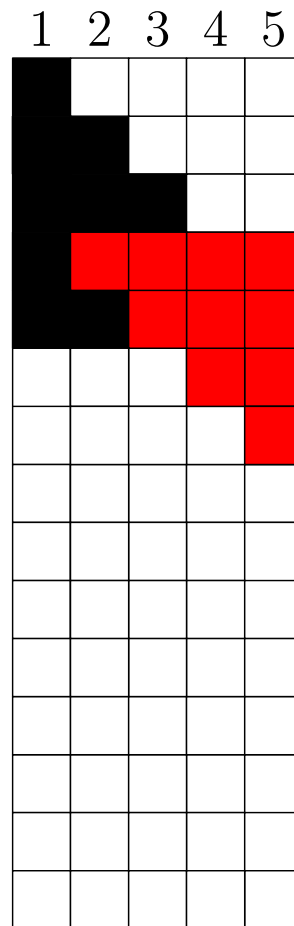
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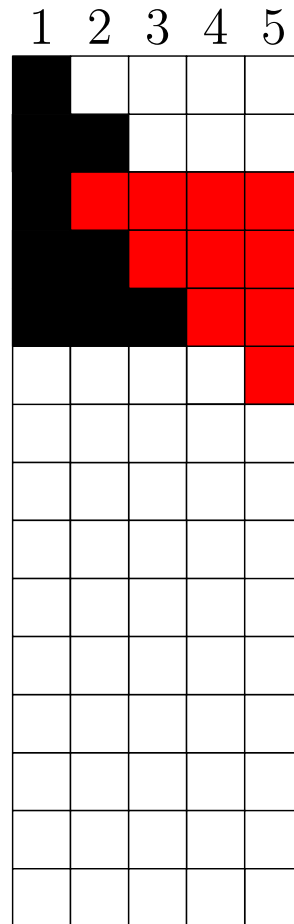
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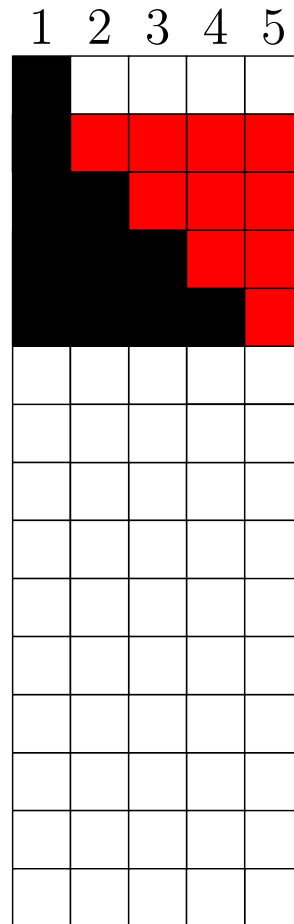
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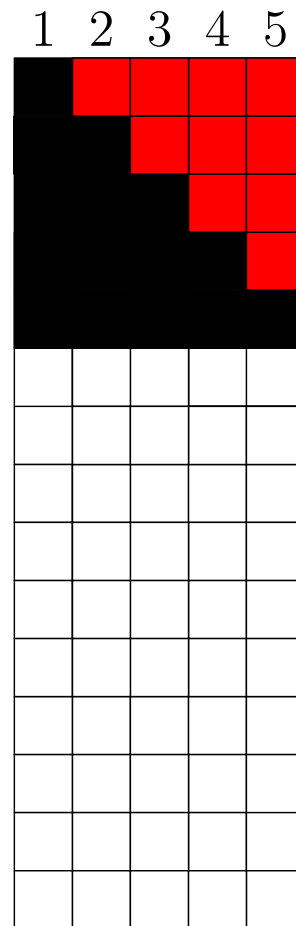
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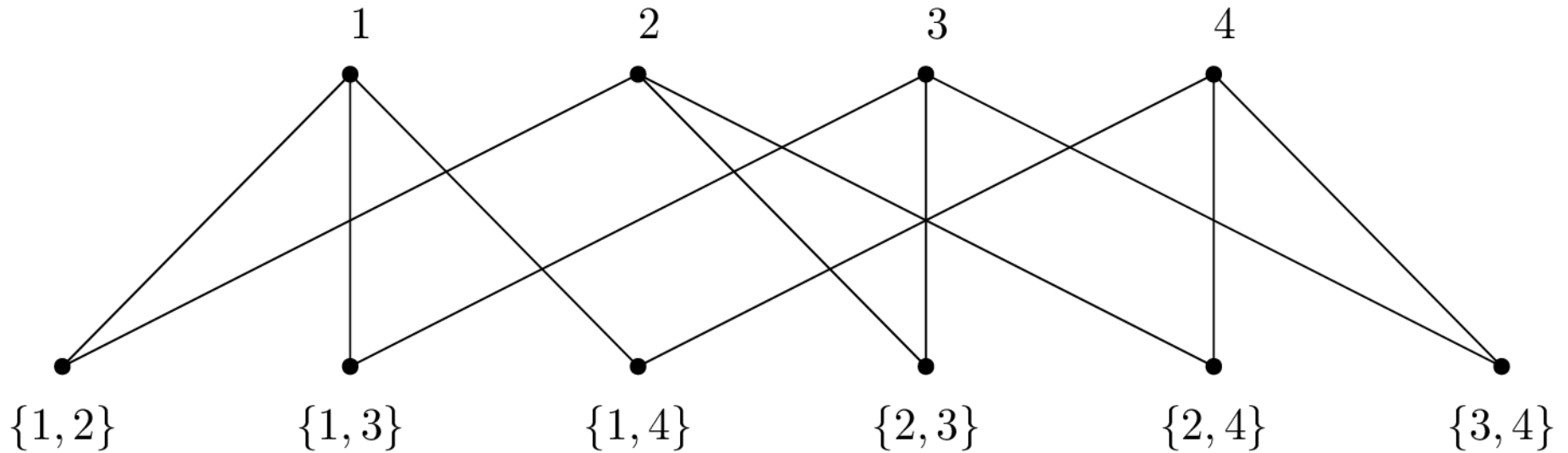
Coloring of T



EFL and b -coloring of tight bipartite graphs

- [Lin, Chang (2013)]:
EFL implies that class B_n of tight bipartite graphs is n - or $(n-1)$ - b -colorable.
- In proof EFL is used as following:
Let G be tight bipartite graph and G^* it's conversion to graph satisfying hypothesis of clique version of EFL.
If G^* is n colorable, then G is n - or $(n-1)$ - b -colorable.

EFL and b-coloring of tight bipartite graphs

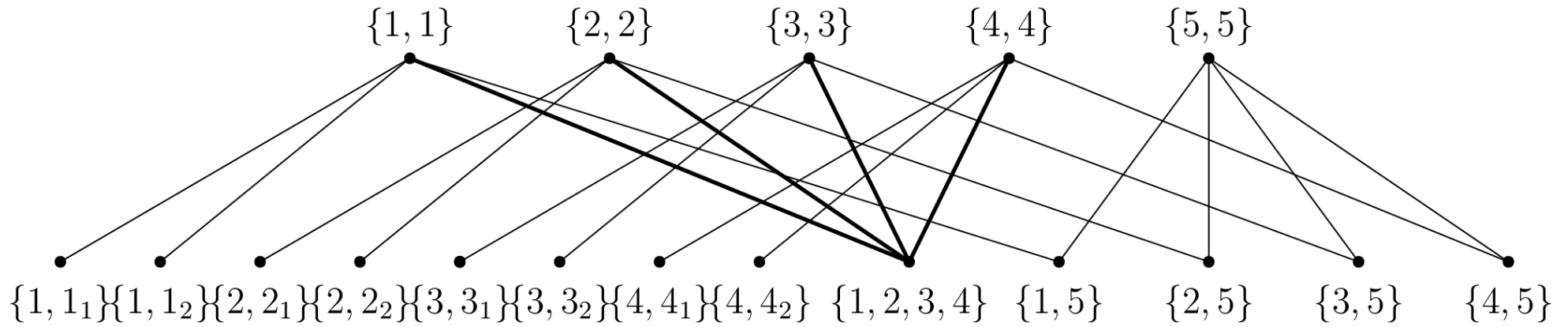


- $S(K_n)$ – graph created from K_n by subdivision of all edges
- Tetris for $S(K_n)^*$ is maximal tetris.

EFL and b-coloring of tight bipartite graphs

- n -b-colorability in tetrises: all bricks with one filled tile have distinct colors
- [Lin, Chang (2013)]:
 - $S(K_n)$ is n -b-colorable for n odd
 - $S(K_n)$ is $(n-1)$ -b-colorable for n even
- Proof in tetrises:
 - n odd: from coloring of maximal tetris
 - n even: $S(K_n)$ not n -b-colorable and EFL holds for $S(K_n)$ from tetris coloring algorithm

EFL and b-coloring of tight bipartite graphs



- $G_{n,k}$ – taking $S(K_n)$, merging vertices $\{1,2\}, \dots, \{1,k\}$ into one, adding some vertices of degree 1
- In tetrises: $G_{n,k}^*$ is tetris with one brick with k filled tiles, rest are all bricks with two filled tiles which can be added (and some bricks with one filled tile).

EFL and b-coloring of tight bipartite graphs

- [Lin, Chang (2013)]: All graphs in $G_{n,k}$ are n - or $(n-1)$ -b-colorable.
- Proof using tetrises:
 - Color tetris of $G_{n,k}^*$ with n colors (alteration of algorithm for tetris with maximal number of crossings)
 - Implies n - or $(n-1)$ -b-colorability by theorem of [Lin, Chang (2013)]

EFL and dense graphs

- $\text{deg}(x)$ – number of cliques in which vertex/brick x belongs
- Two bricks/vertices collide, if they belong to same clique
- [Sánchez-Arroyo (2008)]:

Let k be number of bricks with degree at least $\text{deg}(x)$ colliding with x , then

$$k \leq \frac{\text{deg}(x)}{\text{deg}(x)-1} \cdot (n - \text{deg}(x)) + 1$$

EFL and dense graphs

- New result:
 - Choose any p bricks.
 - Denote z number of cliques containing at least one of these chosen bricks.
 - Let x be any brick with $\deg(x) > p+1$
 - Let k be number of bricks with degree at least $\deg(x)$ coliding with x
 - Then $k \leq \frac{\deg(x)}{\deg(x)-1-p} \cdot (n - \deg(x) - z + p) + 1 + p$
 - Note that for $p = 0$ we get result from [Sánchez-Arroyo (2008)]

EFL and dense graphs

- Let d be minimal degree of brick in tetris T and D maximal degree of a brick in tetris T .
- Corollary:
 - If $d > p+1$ and
$$d \cdot z + (d - p)^2 + p > (p + 1) \cdot n$$
 - Then T can be colored by at most n colors.
- Corollary (weaker):
 - If d is at least 3 and d times D is at least $2n$, then T can be colored by at most n colors.



Conclusion

- We have seen and understood many different approaches to EFL and its connection to other mathematical structures.
- We have proven results about b-coloring of tight bipartite graphs in a different and easier way.
- We have generalized and improved results for dense graphs.

References

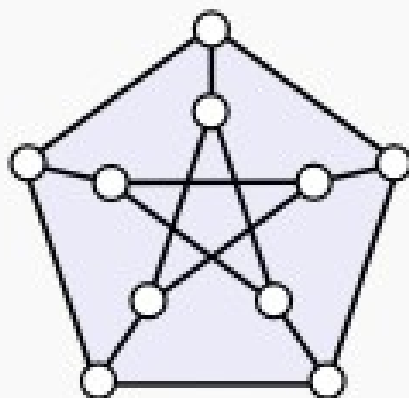
- Wu-Hsiung Lin and Gerard J. Chang. b -coloring of tight bipartite graphs and the Erdős–Faber–Lovász conjecture. *Discrete Applied Mathematics*, 161(7):1060 – 1066, 2013. ISSN 0166-218X.
- Abdón Sánchez-Arroyo. The Erdős–Faber–Lovász conjecture for dense hypergraphs. *Discrete Mathematics*, 308(5): 991 – 992, 2008. ISSN 0012-365X. Selected Papers from 20th British Combinatorial Conference.

Joke

Thanks for watching!

HOW A GRAPH THEORIST DRAWS A "STAR":

FIRST DRAW THE
PETERSEN GRAPH



YES, MY PETERSEN GRAPH
LOOKS THIS GOOD!

NOW ERASE THE OUTSIDE!



ANOTHER PERFECT STAR.

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