Tetrises and Graph Coloring

(joke included)

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Erdős–Faber–Lovász conjecture - clique version

- If $n$ complete graphs, each having exactly $n$ vertices, have the property that every pair of complete graphs has at most one shared vertex, then the union of the graphs can be colored with $n$ colors.

- Example: $n = 5$
Tetrises
New Perspective on EFL
Different perspectives on EFL

- EFL relational structure version
- b-coloring of tight bipartite graphs
- EFL assembling of tetrises
- EFL hypergraph version (edge coloring)
- EFL hypergraph version (vertex coloring)
- EFL clique version
- EFL set version
Tetris with maximal number of crossings $T$

- Each two cliques intersect in one vertex.
- No vertex belongs to more than 2 cliques.
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EFL and b-coloring of tight bipartite graphs

- [Lin, Chang (2013)]: EFL implies that class $Bn$ of tight bipartite graphs is $n$- or $(n-1)$-b-colorable.

- In proof EFL is used as following:
  Let $G$ be tight bipartite graph and $G^*$ it’s conversion to graph satisfying hypothesis of clique version of EFL. If $G^*$ is $n$ colorable, then $G$ is $n$- or $(n-1)$-b-colorable.
EFL and b-coloring of tight bipartite graphs

- $S(K_n)$ – graph created from $K_n$ by subdivision of all edges
- Tetris for $S(K_n)^*$ is maximal tetris.
EFL and b-coloring of tight bipartite graphs

- $n$-b-colorability in tetris: all bricks with one filled tile have distinct colors
- [Lin, Chang (2013)]:
  - $S(K_n)$ is $n$-b-colorable for $n$ odd
  - $S(K_n)$ is $(n-1)$-b-colorable for $n$ even
- Proof in tetris:
  - $n$ odd: from coloring of maximal tetris
  - $n$ even: $S(K_n)$ not $n$-b-colorable and EFL holds for $S(K_n)$ from tetris coloring algorithm
EFL and b-coloring of tight bipartite graphs

- $G_{n,k}$ – taking $S(K_n)$, merging vertices $\{1,2\}, \ldots, \{1,k\}$ into one, adding some vertices of degree 1

- In tetris: $G_{n,k}^*$ is tetris with one brick with $k$ filled tiles, rest are all bricks with two filled tiles which can be added (and some bricks with one filled tile).
EFL and b-coloring of tight bipartite graphs

- [Lin, Chang (2013)]: All graphs in $G_{n,k}$ are $n$- or $(n-1)$-b-colorable.

- Proof using tetris:
  - Color tetris of $G_{n,k}^*$ with $n$ colors (alteration of algorithm for tetris with maximal number of crossings)
  - Implies $n$- or $(n-1)$-b-colorability by theorem of [Lin, Chang (2013)]
EFL and dense graphs

- \( \text{deg}(x) \) – number of cliques in which vertex/brick \( x \) belongs

- Two bricks/vertices collide, if they belong to same clique

- [Sánchez-Arroyo (2008)]:

  Let \( k \) be number of bricks with degree at least \( \text{deg}(x) \) coliding with \( x \), then

  \[
  k \leq \frac{\text{deg}(x)}{\text{deg}(x)-1} \cdot (n - \text{deg}(x)) + 1
  \]
New result:

- Choose any $p$ bricks.
- Denote $z$ number of cliques containing at least one of these chosen bricks.
- Let $x$ be any brick with $\deg(x) > p + 1$
- Let $k$ be number of bricks with degree at least $\deg(x)$ colliding with $x$
- Then $k \leq \frac{\deg(x)}{\deg(x) - 1 - p} \cdot (n - \deg(x) - z + p) + 1 + p$
- Note that for $p = 0$ we get result from [Sánchez-Arroyo (2008)]
EFL and dense graphs

- Let $d$ be minimal degree of brick in tetris $T$ and $D$ maximal degree of a brick in tetris $T$.

- Corollary:
  - If $d > p+1$ and
    
    $$d \cdot z + (d - p)^2 + p > (p + 1) \cdot n$$
  - Then $T$ can be colored by at most $n$ colors.

- Corollary (weaker):
  - If $d$ is at least 3 and $d$ times $D$ is at least $2n$, then $T$ can be colored by at most $n$ colors.
Conclusion

- We have seen and understood many different approaches to EFL and its connection to other mathematical structures.
- We have proven results about b-coloring of tight bipartite graphs in a different and easier way.
- We have generalized and improved results for dense graphs.

HOW A GRAPH THEORIST DRAWS A "STAR":

FIRST DRAW THE PETERSEN GRAPH

YES, MY PETERSEN GRAPH LOOKS THIS GOOD!

NOW ERASE THE OUTSIDE!

ANOTHER PERFECT STAR.