FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

Tetrises and Graph Coloring

(joke included)

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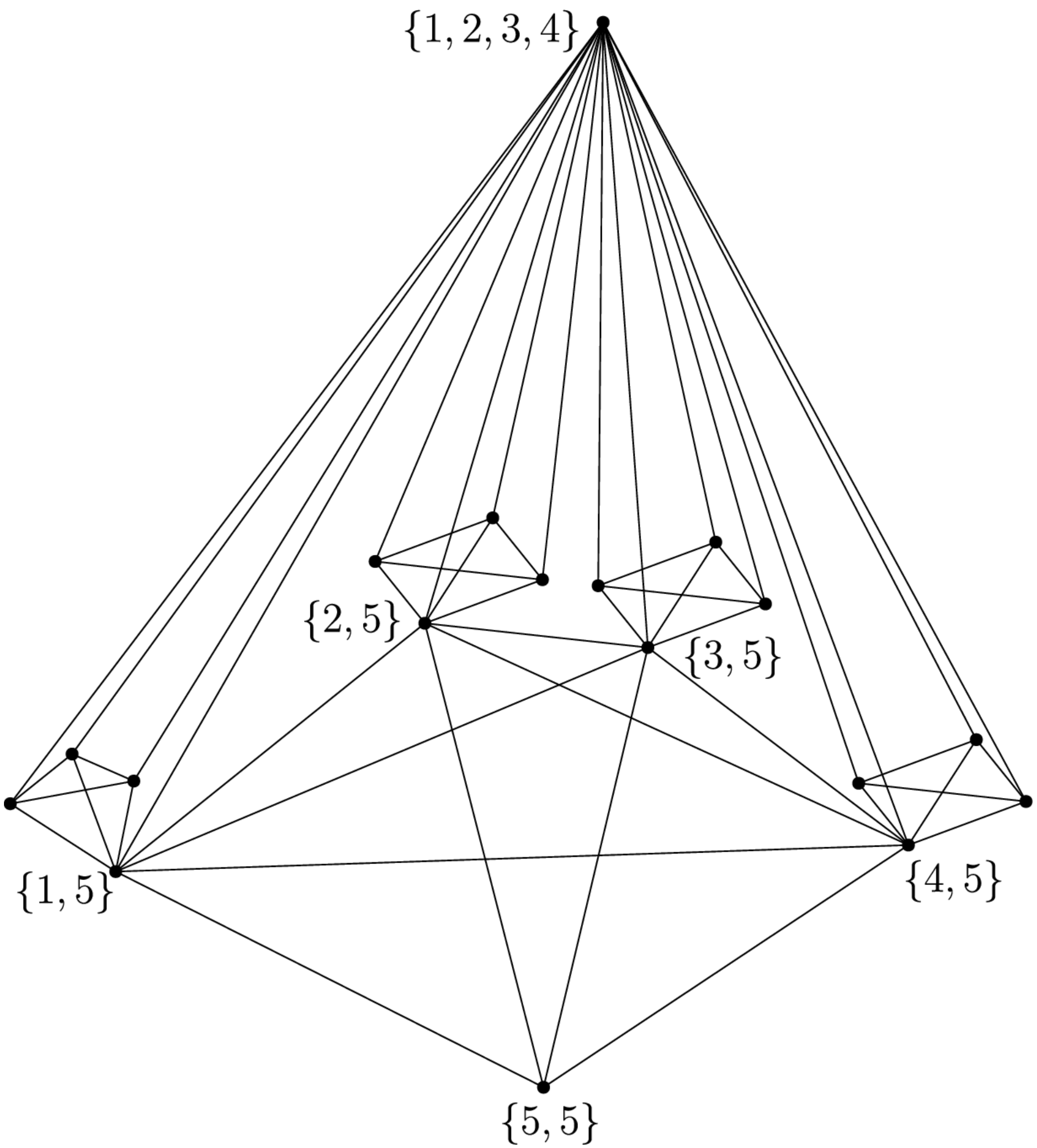
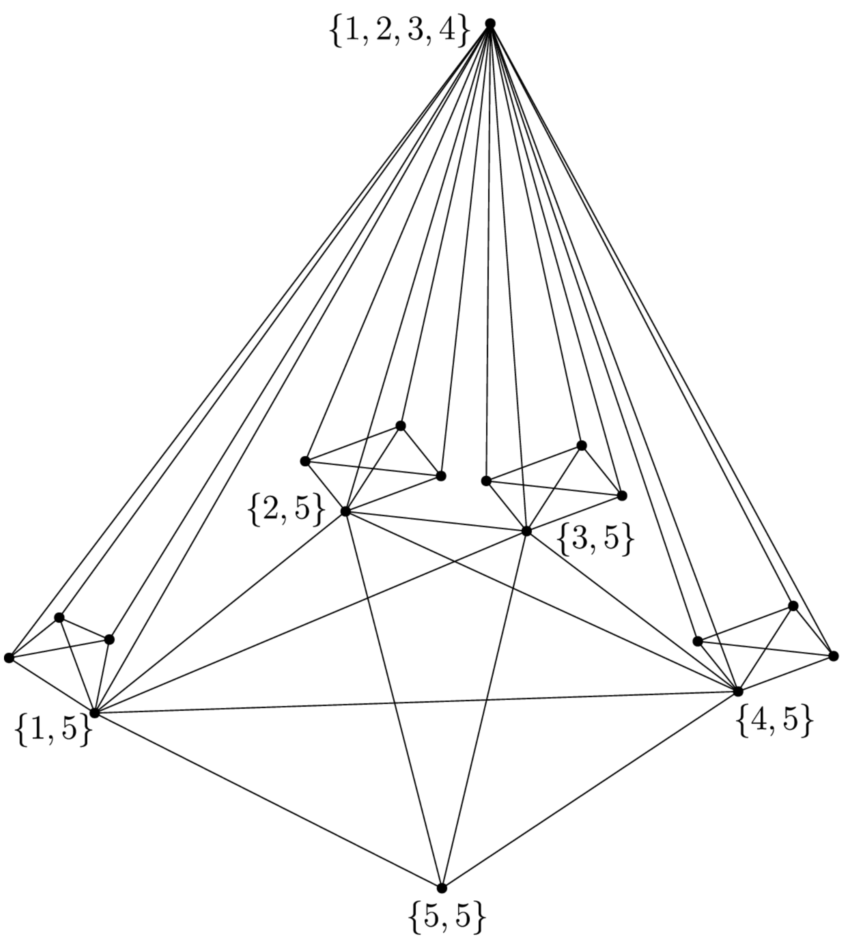
Erdős–Faber–Lovász conjecture - clique version

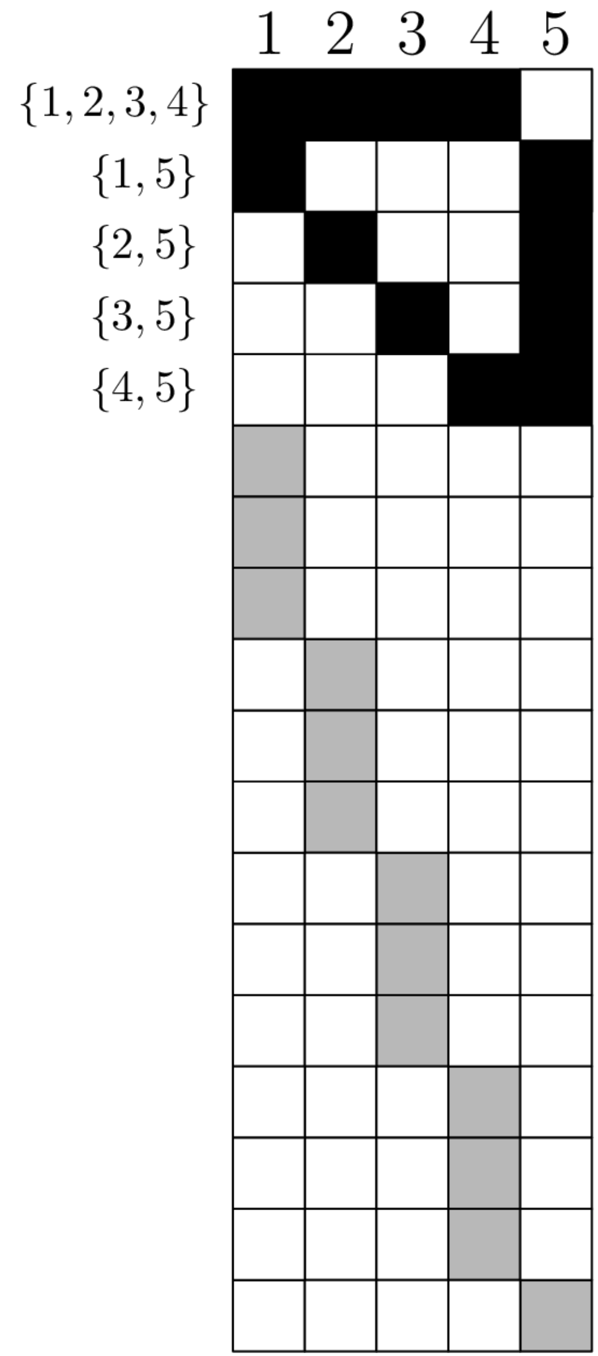
- If n complete graphs, each having exactly n vertices, have the property that every pair of complete graphs has at most one shared vertex, then the union of the graphs can be colored with n colors.
- Example: n = 5



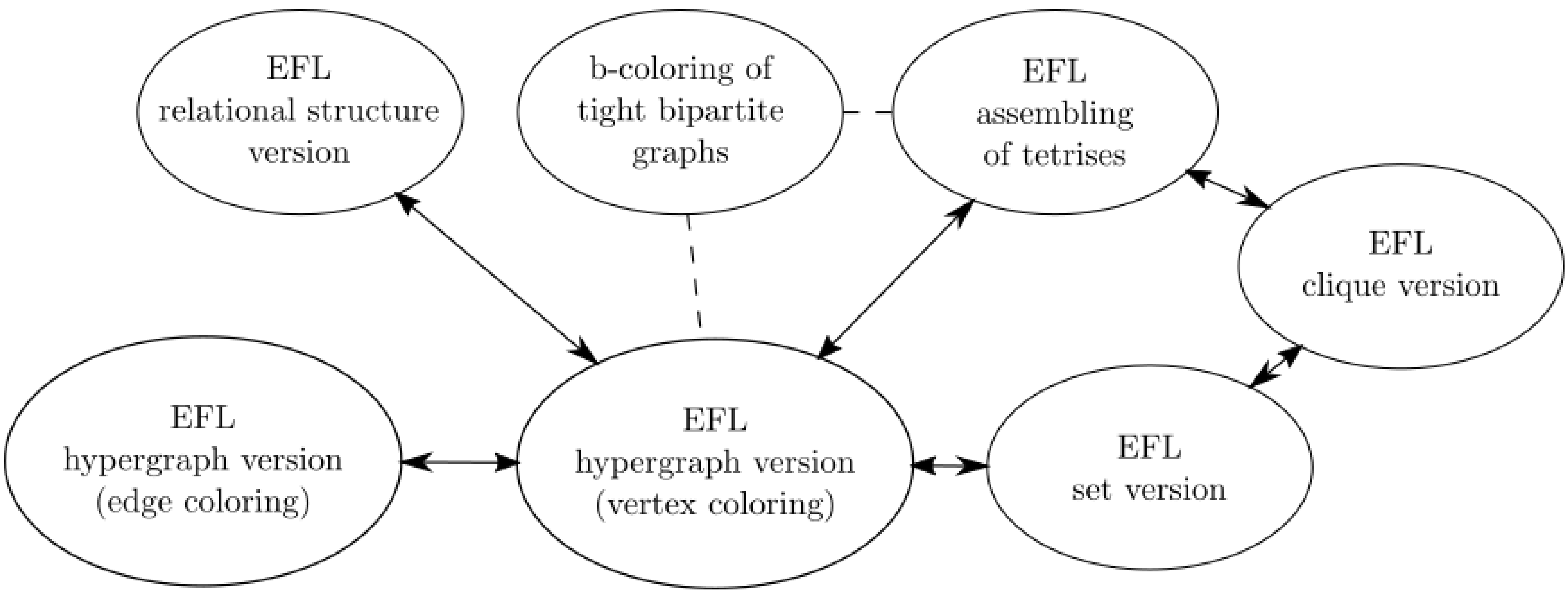
Tetrises

New Perspective on EFL



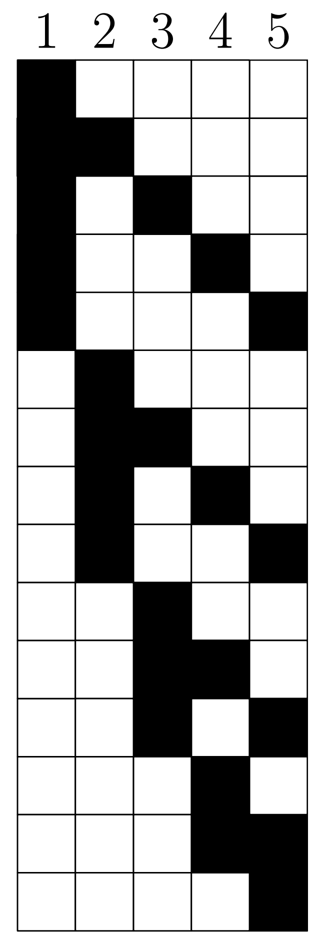


Different perspectives on EFL

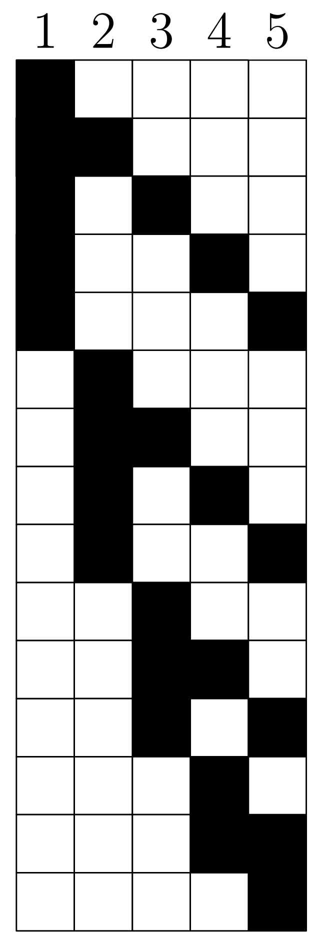


Tetris with maximal number of crossings T

- Each two cliques intersect in one vertex.
- No vertex belongs to more than 2 cliques.



Coloring of T



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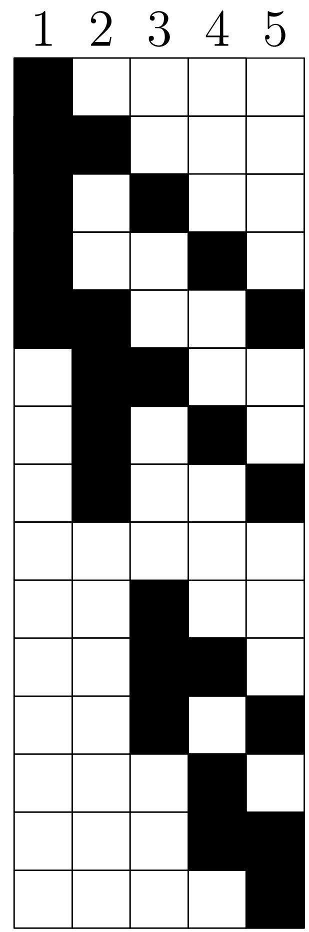

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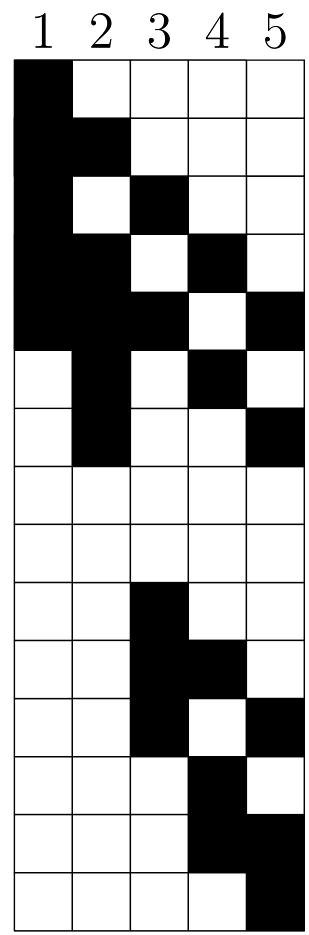

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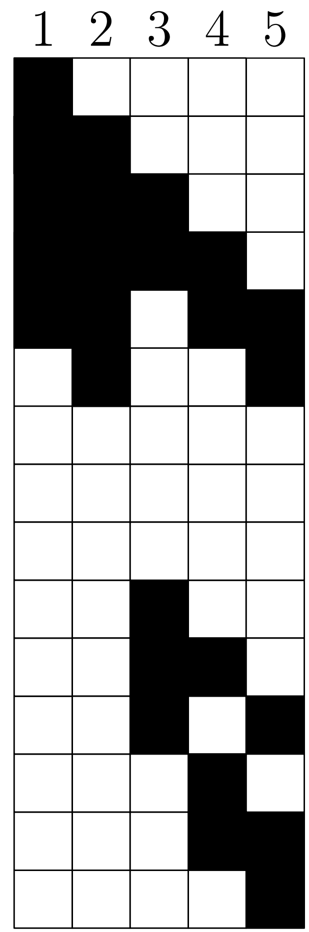

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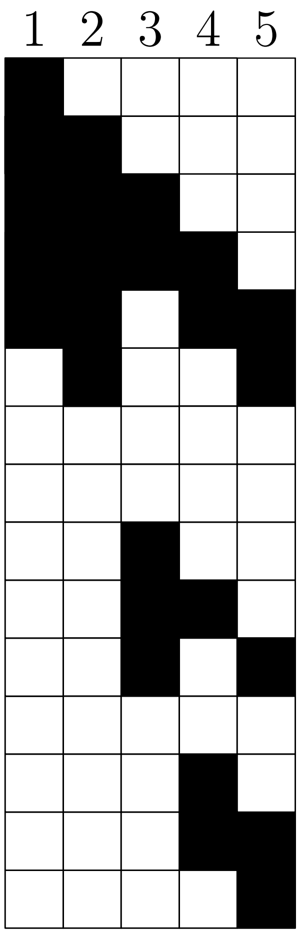

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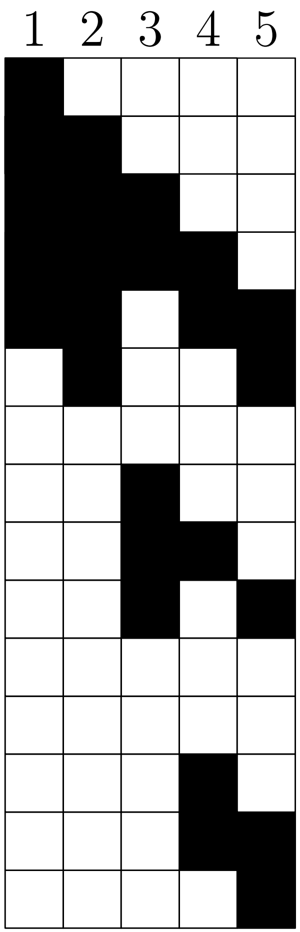

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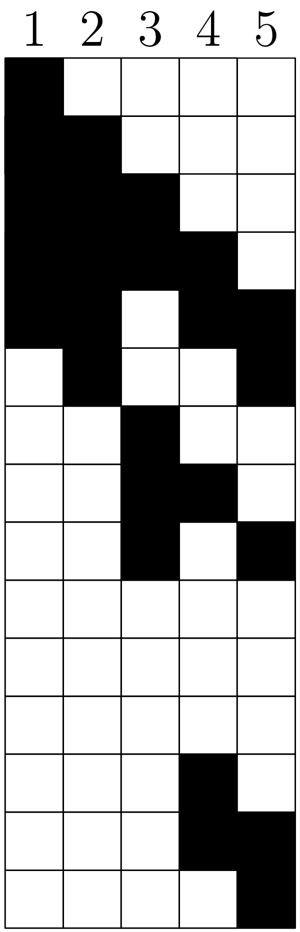

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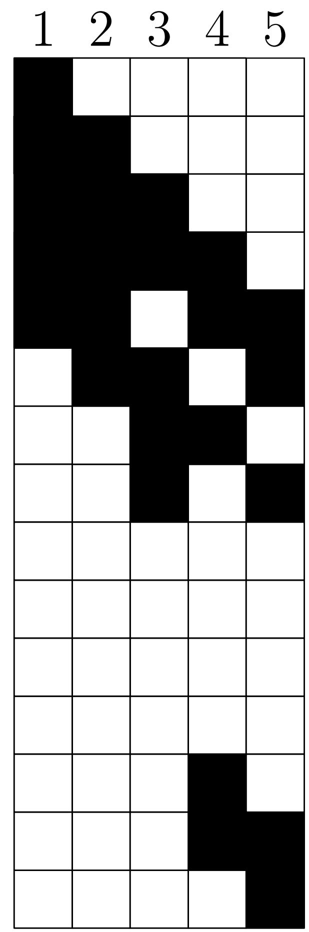

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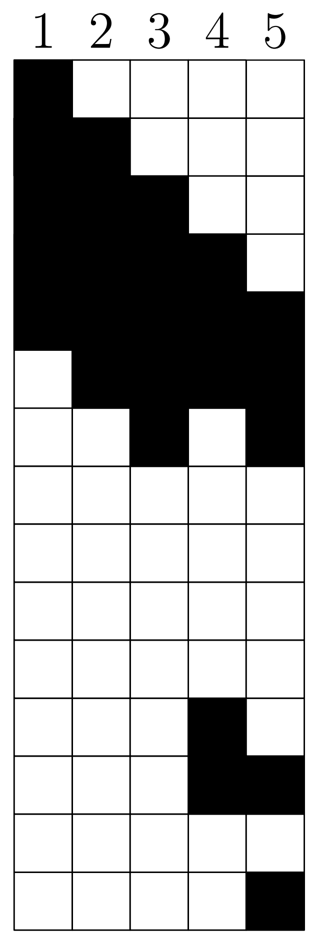

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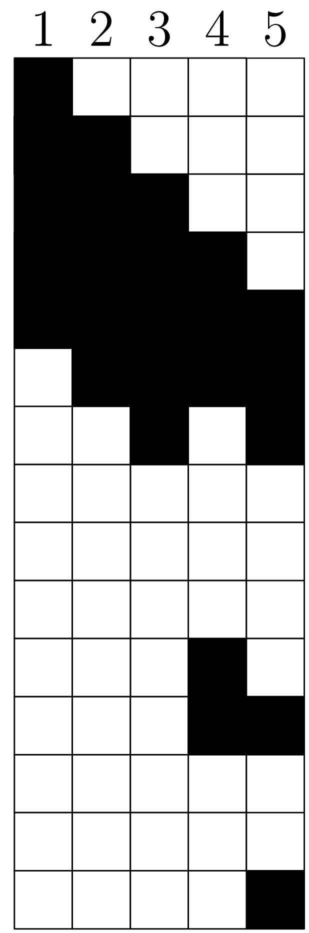

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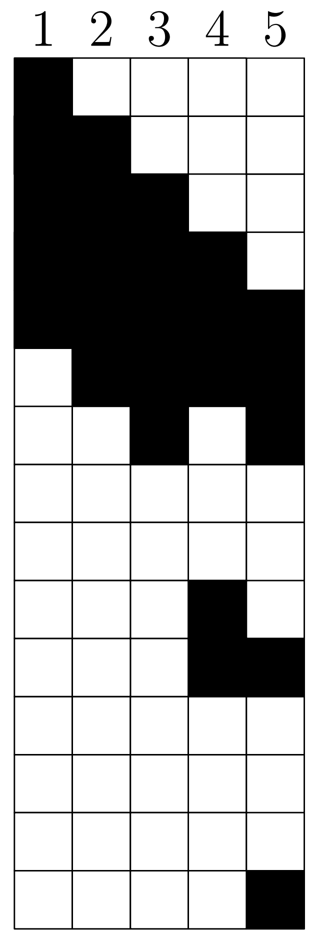

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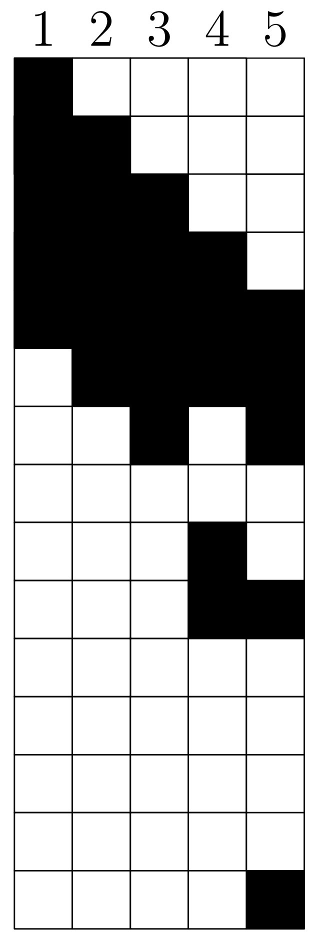

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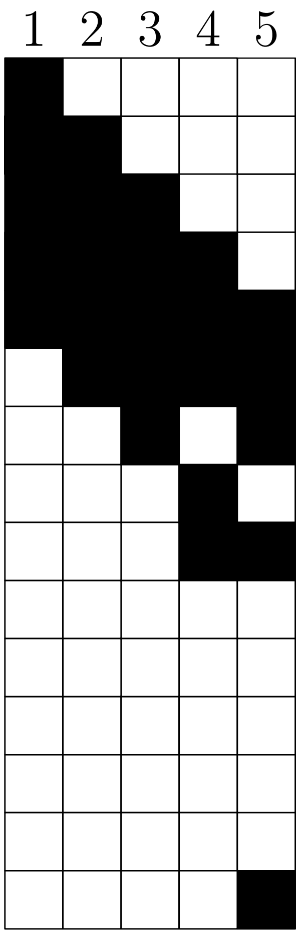

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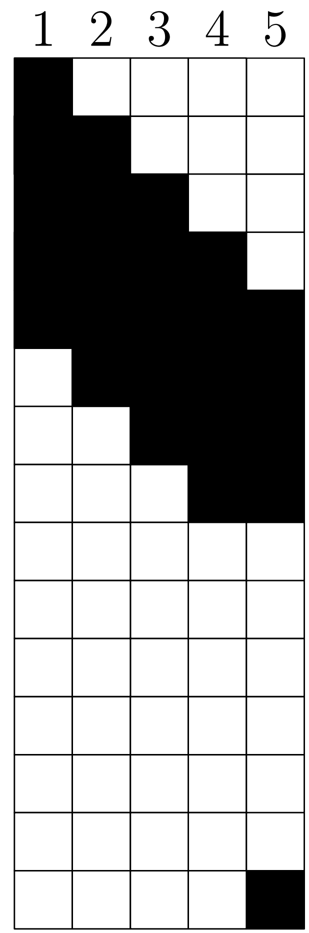

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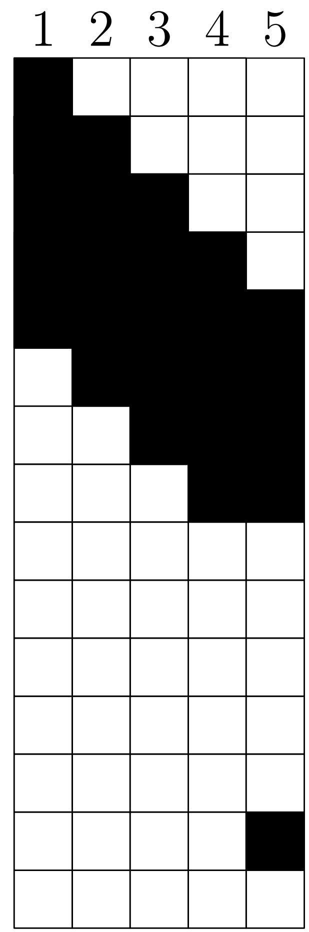

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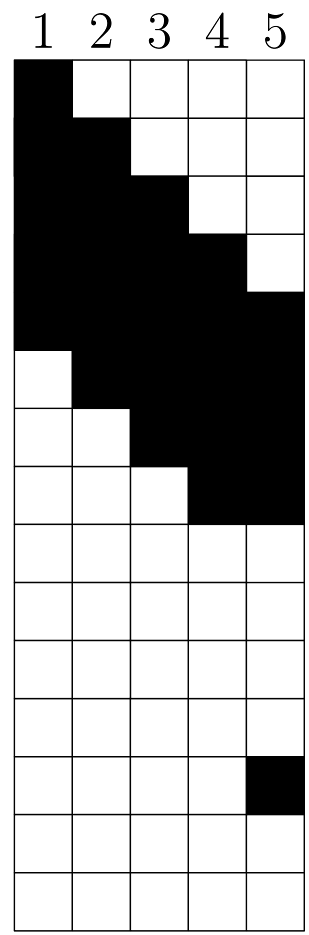

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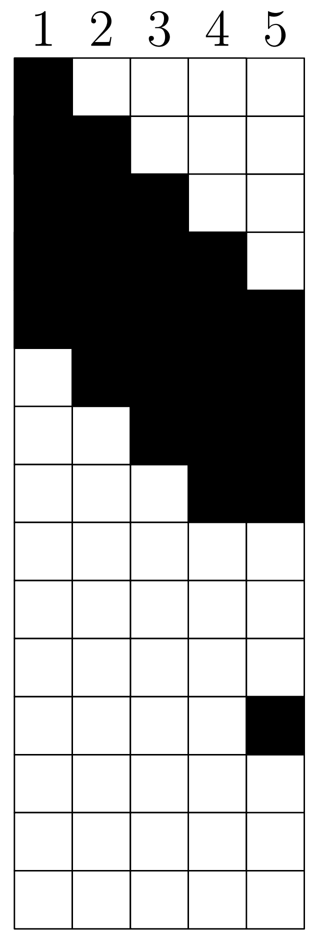

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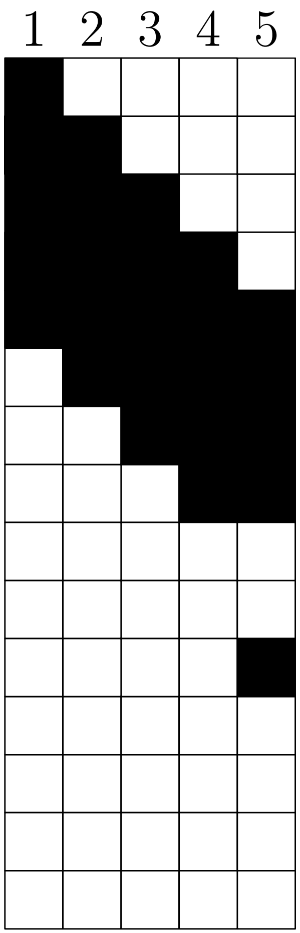

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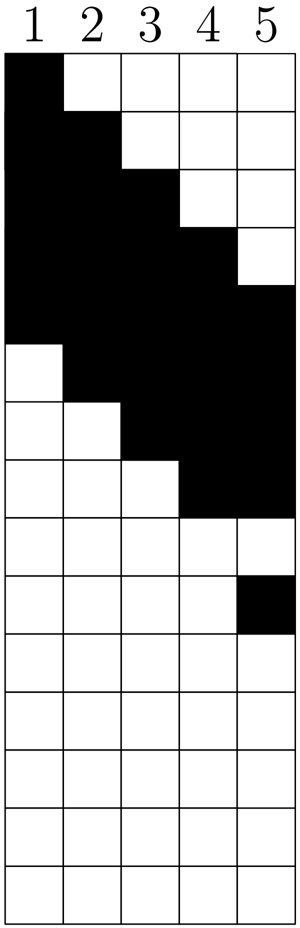

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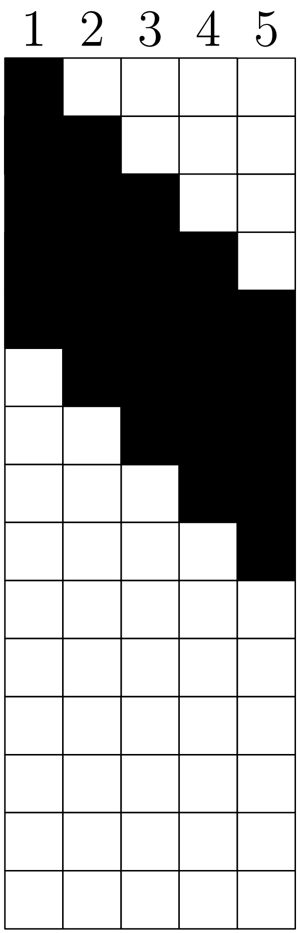

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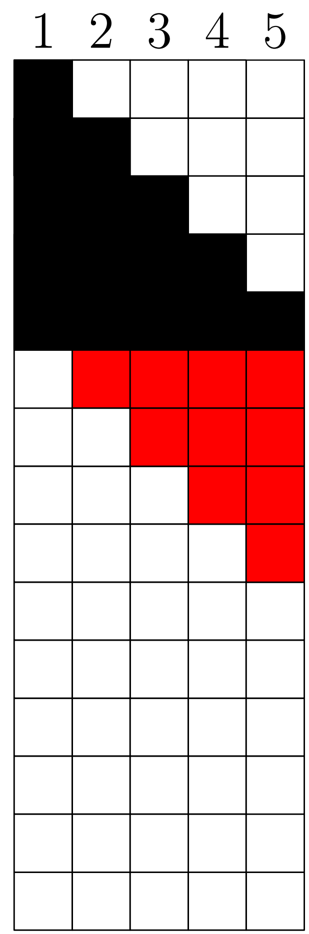

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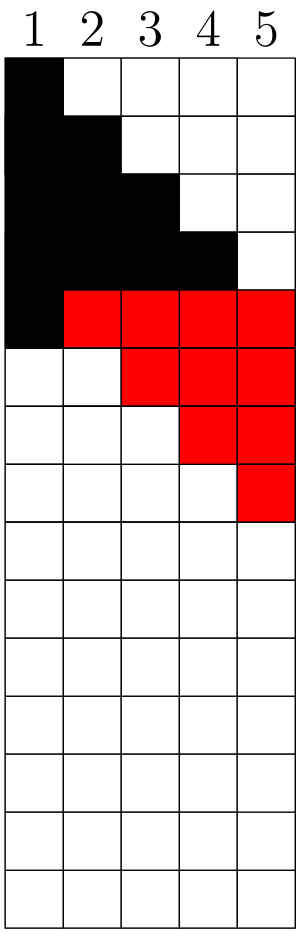

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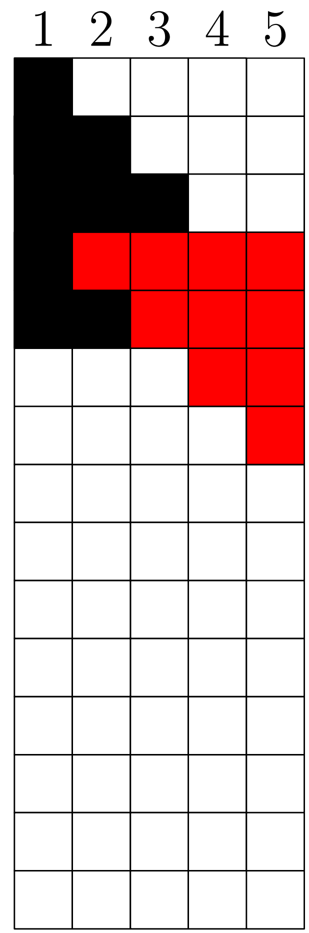

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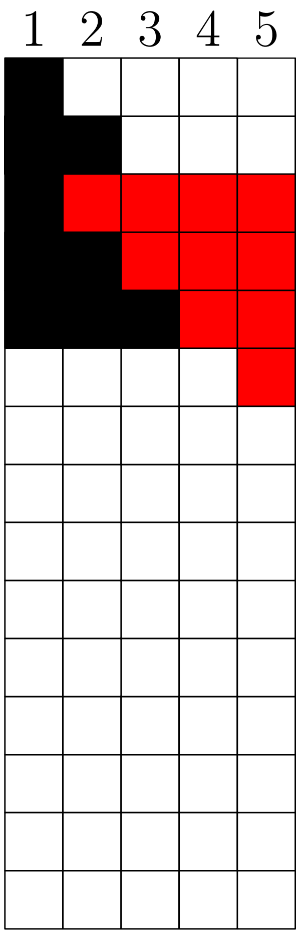

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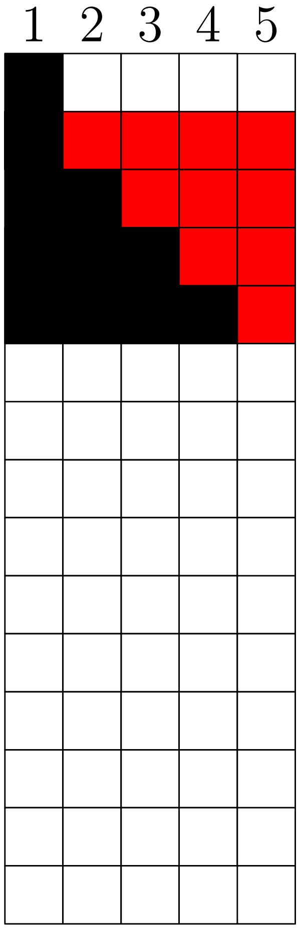

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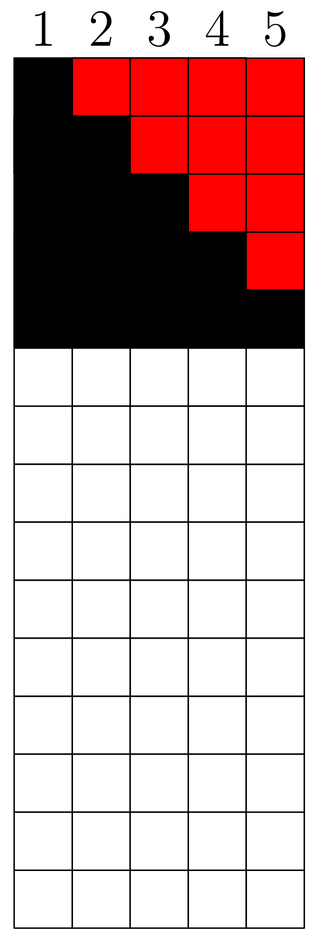
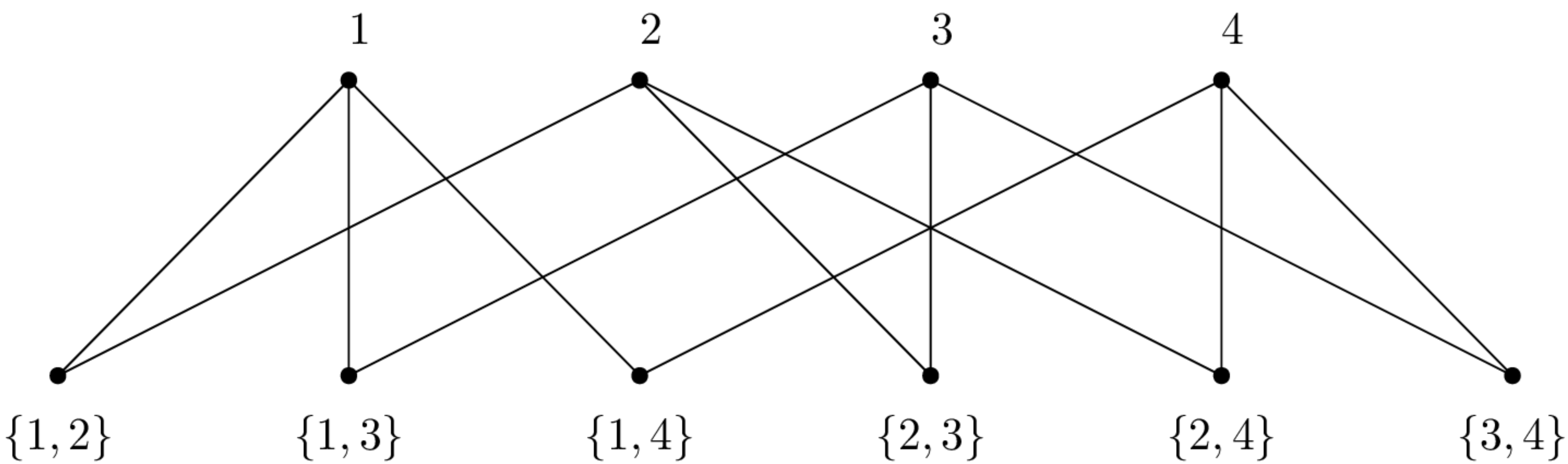


EFL and b-coloring of tight bipartite graphs

- [Lin, Chang (2013)]:
  EFL implies that class *Bn* of tight bipartite graphs is *n*- or *(n-1)-b*-colorable.
- In proof EFL is used as following:

  Let *G* be tight bipartite graph and *G** it's conversion to graph satisfying hypothesis of clique version of EFL. If *G** is *n* colorable, then *G* is *n*- or *(n-1)-b*-colorable.

EFL and b-coloring of tight bipartite graphs

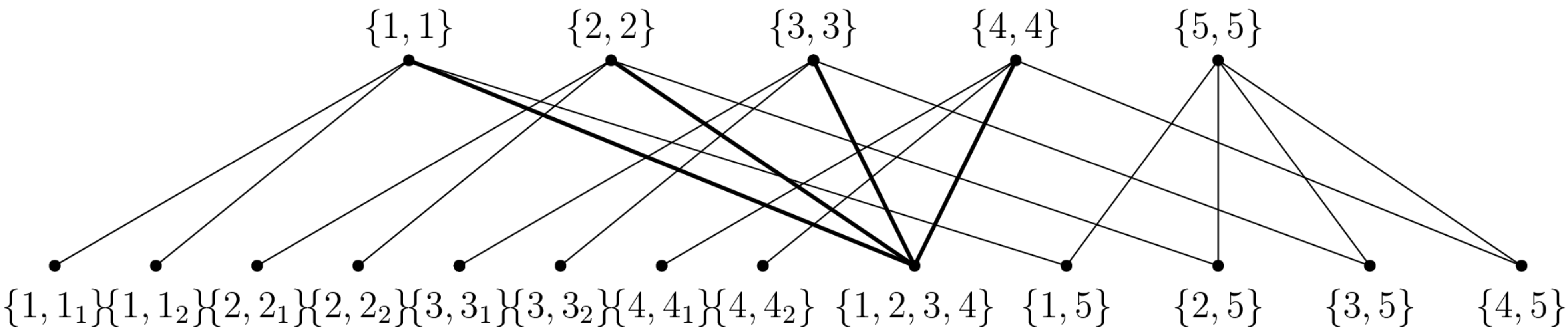


- S(Kn) – graph created from Kn by subdivision of all edges
- Tetris for S(Kn)* is maximal tetris.

EFL and b-coloring of tight bipartite graphs

- *n*-b-colorability in tetrises: all bricks with one filled tile have distinct colors
- [Lin, Chang (2013)]:
  - S(Kn) is *n*-b-colorable for *n* odd
  - S(Kn) is *(n-1)*-b-colorable for *n* even
- Proof in tetrises:
  - *n* odd: from coloring of maximal tetris
  - *n* even: S(Kn) not *n*-b-colorable and EFL holds for S(Kn) from tetris coloring algorithm

EFL and b-coloring of tight bipartite graphs



- Gn,k – taking S(Kn), merging vertices {1,2}, … , {1,*k*} into one, adding some vertices of degree 1
- In tetrises: Gn,k* is tetris with one brick with *k* filled tiles, rest are all bricks with two filled tiles which can be added (and some bricks with one filled tile).

EFL and b-coloring of tight bipartite graphs

- [Lin, Chang (2013)]: All graphs in Gn,k are *n*- or *(n-1)*-b-colorable.
- Proof using tetrises:
  - Color tetris of Gn,k* with *n* colors (alteration of algorithm for tetris with maximal number of crossings)
  - Implies *n*- or *(n-1)*-b-colorability by theorem of [Lin, Chang (2013)]

EFL and dense graphs

- deg(x) – number of cliques in which vertex/brick x belongs
- Two bricks/vertices collide, if they belong to same clique
- [Sánchez-Arroyo (2008)]:

  Let *k* be number of bricks with degree at least deg(x) coliding with x, then

$$k \leq \frac{\deg(x)}{\deg(x)-1} \cdot (n - \deg(x)) + 1$$

EFL and dense graphs

- New result:
  - Choose any *p* bricks.
  - Denote *z* number of cliques containing at least one of these chosen bricks.
  - Let *x* be any brick with deg(x) > *p+1*
  - Let *k* be number of bricks with degree at least deg(x) coliding with *x*
  - Then $k \leq \frac{\deg(x)}{\deg(x)-1-p} \cdot (n - \deg(x) - z + p) + 1 + p$
  - Note that for $p = 0$ we get result from [Sánchez-Arroyo (2008)]

EFL and dense graphs

- Let d be minimal degree of brick in tetris T and *D* maximal degree of a brick in tetris T.
- Corollary:
  - If d > p+1 and
    $$d \cdot z + (d-p)^2 + p > (p+1) \cdot n$$
  - Then T can be colored by at most *n* colors.
- Corollary (weaker):
  - If *d* is at least 3 and *d* times *D* is at least *2n*, then T can be colored by at most *n* colors.

Conclusion

- We have seen and understood many different approaches to EFL and it's connection to other mathematical structures.
- We have proven results about b-coloring of tight bipartite graphs in a different and easier way.
- We have generalized and improved results for dense graphs.

References

- Wu-Hsiung Lin and Gerard J. Chang. b-coloring of tight bipartite graphs and the Erdős–Faber–Lovász conjecture. Discrete Applied Mathematics, 161(7):1060 – 1066, 2013. ISSN 0166-218X.
- Abdón Sánchez-Arroyo. The Erdős–Faber–Lovász conjecture for dense hypergraphs. Discrete Mathematics, 308(5): 991 – 992, 2008. ISSN 0012-365X. Selected Papers from 20th British Combinatorial Conference.

Joke

Thanks for watching!