Does positivity determine quantum cohomology?

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1. Backgrounds and Terminology
2. The problem and previous results

1 Backgrounds and Terminology

- Construction of Grassmannians
- Schubert Varieties
- Schubert Calculus
- Quantum Cohomology

Construction of Grassmannians

Definition 1.1 *Let* $k = \overline{k}$ be algebraically closed field, and let k^n be the vector space of column vectors with n coordinates. Give a non-negative integer $m \leq n$, the **Grassmannian variety** $\boldsymbol{Gr(m,n)}$ is defined as a set by

$$Gr(m,n) = \{\Sigma \in k^n | \Sigma \text{ is a vector subspace with } \dim(\Sigma)=m\}.$$

Definition 1.2 *A* ***Schubert symbol*** *for* $Gr(m,n)$ *is any subset* I of cardinally m in the integer interval $[1,n]$. For any Schubert symbol I, we set $\Sigma_I = Span\{e_i | i \in I\}$.

Schubert Varieties

➢ Schubert Varieties

Definition 1.3 Let $F_\bullet = (F_1 \subset F_2 \subset \cdots \subset F_n)$ be the standard flag of $V = \mathrm{k}^n$, where $F_i = Span\{e_1, e_2, \dots, e_i\}$. Given a point $\Sigma \in X$, define an **associated Schubert symbol** by

$$I(\Sigma) = \{i \in [1, n] | \Sigma \cap F_i \supset \Sigma \cap F_{i-1}\}.$$

Moreover, for any Schubert symbol J for X we set

$$\dot{X}_J = \{\Sigma \in X | I(\Sigma) = J\},$$

which is called **Schubert cells**.

⬇

Their closures are called Schubert varieties.

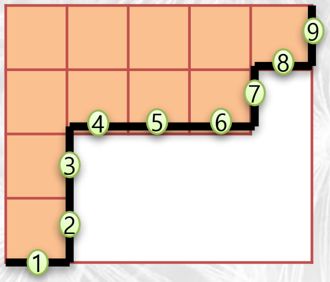
Schubert Varieties

Young Diagrams

Each Schubert symbols for $X = Gr(m, n)$ corresponds to a Young diagram of boxes contained in the rectangle with m rows and $n - m$ columns. Consider the path from the lower-left corner to the upper-right corner consisting n steps, where

- i-th step is vertical if $i \in I$ and
- i-th step is horizontal if $i \notin I$.

Example 1.4 The Schubert symbol $I = \{2,3,7,9\}$ for $Gr(4,9)$ corresponds to the following Young diagram:



If λ is the Young diagram corresponding to J, then we will denote the Schubert variety X_J also by X_λ.

Schubert Calculus

➢ Homology and Cohomology

Definition 1.5 *A* ***chain complex*** *is a sequence*

$$\cdots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

of chain groups together with boundary homomorphisms ∂_i between the chain groups with the important property that $\partial_i \partial_{i+1} = 0$ for all $\boldsymbol{i}$.

implies

$$\mathrm{Im}\partial_{n+1} \subset ker\partial_n$$

We now define the $\boldsymbol{n}$**th homology group** in the chain complex to be the quotient

$$H_n(C) = ker\partial_n / Im\partial_{n+1}.$$

Schubert Calculus

➢ Homology and Cohomology

Definition 1.6 *Given a chain complex*

$$\cdots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

*and a group G, we can define the **cochains** C_n^* to be the respective groups of all homomorphisms from C_n to G:*

$$C_n^* = Hom(C_n, G).$$

*We define the **coboundary map** $\delta_n: C_{n-1}^* \to C_n^*$ dual to ∂_n as the map sending $\varphi \mapsto \delta_n \varphi = \partial_n^* \varphi$.*

We now define the ***n*th cohomology group** as the quotient:

$$H^n(C; G) = ker\delta_{n+1}/Im\delta_n.$$

Schubert Calculus

➢ Cohomology Ring H*(X)

$H^*(X)$

The Schubert classes $[X^I]$ *form an* ***additive basis***

The ***multiplicative structure*** *is defined by*

$$[X^I] \cdot [X^J] = \sum_K C_{I,J}^K [X^K],$$

where $C_{I,J}^K$ is called Little wood-Richardcon coefficients.

Schubert Calculus

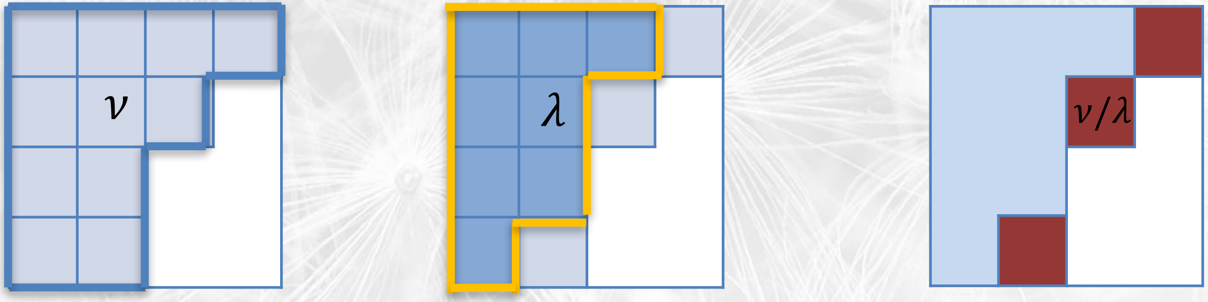
➢ Cohomology Ring H*(X)

➢ The Pieri Formula

Given Young diagrams $\lambda \subset \nu$, let ν/λ denote the skew diagram of boxes in ν that are not in λ. This diagram is called **horizontal strip** if each column contains at most one box.

We identify a Young diagram $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$, where λ_i is the number of boxes in i-th row.

For example, let $\nu = (4,3,2,2)$ in $Gr(4,8)$ and let $\lambda = (3,2,2,1)$, we then have ν/λ is horizontal strip since



Schubert Calculus

➢ Cohomology Ring H*(X)

➢ The Pieri Formula

Theorem 1.7 *Let λ be a Young diagram for X and $p \in [0, n-m]$. Then we have*

$$[X^p] \cdot [X^\lambda] = \sum_\nu [X^\nu],$$

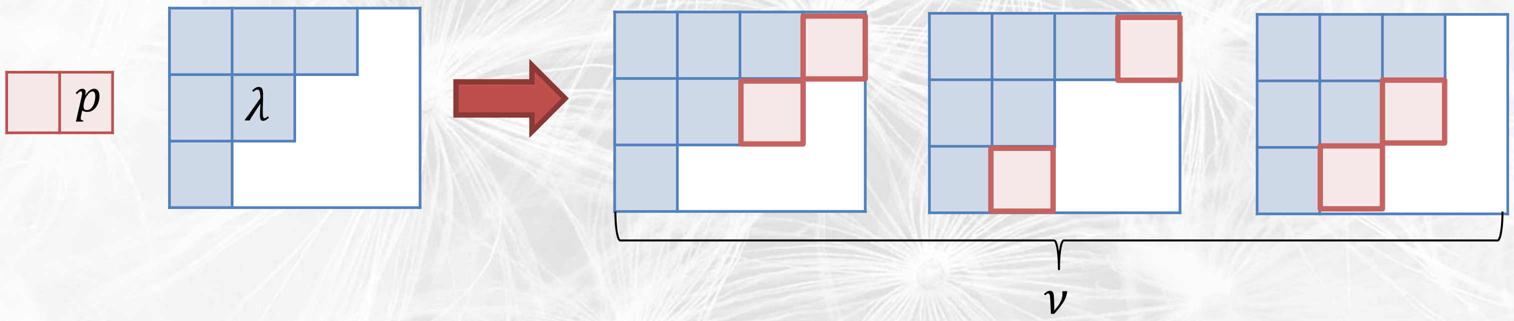
where the sum is over all Young diagram ν for which ν/λ is a horizontal strip of p boxes.

Schubert Calculus

➢ Cohomology Ring H*(X)

➢ The Pieri Formula

For example, let $\lambda = (3,2,1)$ in $Gr(3,7)$ and let $\mathrm{p} = 2$, consider



Then, we have $[X^2] \cdot \left[X^{(3,2,1)}\right] = \left[X^{(4,3,1)}\right] + \left[X^{(4,2,2)}\right] + \left[X^{(3,3,2)}\right]$.

Quantum Cohomology

➢ The small quantum ring

Definition 1.8 *Given Young diagrams λ, μ, ν, and a degree d for which $|\lambda| + |\mu| = |\nu| + nd$, define the* ***Gromov-Witten invariant*** $\langle [X^\lambda], [X^\mu], [X_\nu] \rangle_d$ *to be the number of rational curves $C \subset X$ of degree d that meet the Schubert varieties X^λ, $g \cdot X^\mu$, and X_ν.*

The **small quantum ring** is defined by

$$QH(X) = H^*(X) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$$

which the multiplicative structure is defined by

$$[X^\lambda] \star [X^\mu] = \sum_{\nu, d \geq 0} \langle [X^\lambda], [X^\mu], [X_\nu] \rangle_d \, q^d [X^\nu]$$

where the sum is over all Young diagram ν and degree d.

Quantum Cohomology

➢ The small quantum ring

➢ The quantum Pieri formula

Given Young diagram λ , *let* $\hat{\lambda}$ denote a partition obtained by removing the top row and leftmost column of λ.

Theorem 1.9 *If λ is contained in an* $\mathrm{m}\times(n-m)$ *rectangle and* $p \in [1, n-m]$ *then*

$$[X^p] \star [X^\lambda] = \sum [X^\mu] + q \sum [X^\nu]$$

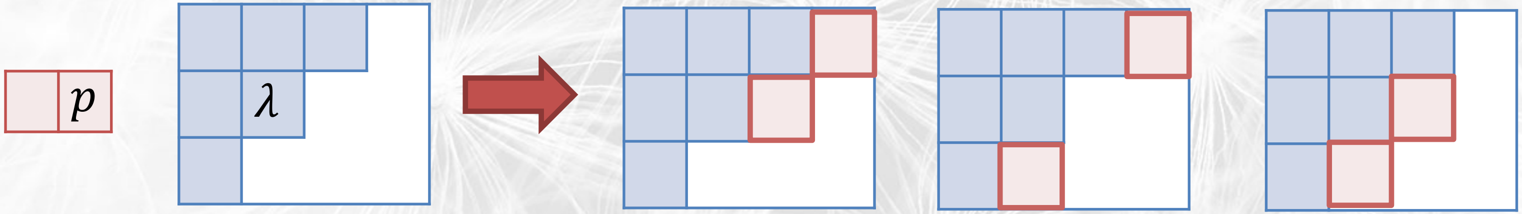
where the first sum is over all partitions μ *s.t.* $\mu\ /\ \lambda$ *is horizontal strip of* p *boxes, and the second sum is over all Young diagram* ν *containing* $\widehat{\lambda}$ *s.t.* $\nu\ /\widehat{\lambda}$ *is horizontal strip,* $\nu_1 < \lambda_1$*, and* $|\nu| = |\lambda| + p - n$.

Schubert Calculus

➢ The small quantum ring

➢ The quantum Pieri formula

For example, let $\lambda = (3,2,1)$ in $Gr(3,7)$ and let $\mathrm{p} = 2$, consider



$$|\nu| = |\lambda| + p - n = 6 + 2 - 7 = 1$$

Then, we have $[X^2] \star \left[X^{(3,2,1)}\right] = \left[X^{(4,3,1)}\right] + \left[X^{(4,2,2)}\right] + \left[X^{(3,3,2)}\right] + \boxed{q\left[X^{(1)}\right].}$

Problem and previous results

- Does positivity determine quantum cohomology?
- Previous results

Does positivity determine quantum cohomology?

- The conjecture of Fomin, Gelfand, and Postnikov

Positivity properties

determine

The small quantum cohomology ring of variety of complete flags $GL(n)/B$

Grassmannian?

Does positivity determine quantum cohomology?

Gromov-Witten invariants
& Schubert Varieties

Facts

Nonnegative integer

Define

Force?

Quantum Cohomology Rings

Facts

Conjecture1

Does positivity determine quantum cohomology?

Conjecture1. *Let QH be any graded $\mathbb{Z}[q]$-algebra that has a $\mathbb{Z}[q]$-basis $\{\tau_\lambda\}$ indexed by Young diagrams λ contained in $m \times (n-m)$, such that $\deg(\tau_\lambda) = |\lambda|$ and $\deg(q) = n$. Assume that the $\mathbb{Z}$-basis $\{q^d \tau_\lambda\}$ multiplies with non-negative structure constants, and the map $\tau_\lambda \longmapsto [X^\lambda]$ defines an isomorphism of ring $QH/\langle q \rangle \cong H^*(X;\mathbb{Z})$. Then there exists a unique $\alpha \in \mathbb{N}$ such that multiplication in QH is determined by*

$$\tau_\lambda \tau_\mu = \sum_{\nu, d \geq 0} \langle X^\lambda, X^\mu, X_\nu \rangle_d (\alpha q)^d \tau_\nu \,.$$

Previous Results

Conjecture1 is true for $Gr(2, n)$

Is it true for $Gr(k, n)$ for $k > 2$?

THANK YOU!

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