Fake Projective Planes

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- There are exactly 100 FPPs, and they come in complex-conjugate pairs (50 pairs in total). Each FPP is realized in the form of B^2/Γ, where $B^2 \subset \mathbb{C}^2$ is the open unit ball, and Γ is some group of biholomorphisms acting on B^2.

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- It is known that every FPP is an algebraic surface (follows from Chow's theorem). However, the above construction only gives the FPPs as a complex manifold; it does not give rise to explicit polynomial equations.
- Out of the 50 pairs of FPPs, only 12 have known equations. The goal for this project is to find explicit polynomial equations for three more pairs of FPPs.

Brief Introduction to Line Bundles

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- **Fact:** The collections of all sections of a bundle W on X forms a finite dimensional vector space over $\mathbb{C}$, which we will denote as $H^0(X, W)$.
- **Fact:** Every complex surface has a *canonical (line) bundle*, which we denote as K.
- **Fact:** For well enough behaved line bundles W on a surface X, choosing a basis for $H^0(X, W)$ gives us a map from X into $\mathbb{CP}^n$, where $n + 1 = \dim H^0(X, W)$.

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So in summary, we find equations in two steps:

1. Find a suitable basis of $H^0(FPP, 2K)$.
2. Look for polynomial relations amongst the basis elements.

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Big Picture

- We started with known equations for an FPP. We will denote this FPP as $\widehat{FPP}$. We used these equations as a starting point to work towards finding equations of three unknown FPPs, denoted FPP_1, FPP_2, and FPP_3, by working through a common covering space as shown in the following diagram:

$$
\begin{tikzcd}
 & & 18FPP \arrow[dll] \arrow[dd] \arrow[ddr] \arrow[dr] & & \\
 & & & 9\widehat{FPP} \arrow[dr] & \\
FPP_3 & & FPP_2 & FPP_1 & \widehat{FPP}
\end{tikzcd}
$$

(here, $9\widehat{FPP}$ is a 9-fold cover of $\widehat{FPP}$, and $18FPP$ is an 18-fold cover of FPP_1, FPP_2, FPP_3, and $\widehat{FPP}$).

Results: Nonreduced Linear Cut

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- We found it by doing an exhaustive search over $\mathbb{F}_{17}\mathbb{P}^9$ and then lifting the solution to $\mathbb{Q}(\sqrt{-2})$.

Results: Nonreduced Linear Cut

For those curious, this was the equation:

$$
\begin{aligned}
Y_0 &+ \frac{772 + 230i\sqrt{2}}{1539} Y_1 + \frac{490092282515 + 172394599619i\sqrt{2}}{74626445502} Y_2 \\
&+ \frac{-103321297235596 - 53009187345521i\sqrt{2}}{8291827278} Y_3 \\
&+ \frac{-231232791152 - 281704162355i\sqrt{2}}{74626445502} Y_4 \\
&+ \frac{-129148133758 - 116323978837i\sqrt{2}}{51044488723368} Y_5 \\
&+ \frac{-442659985194376532 - 470633182584423089i\sqrt{2}}{134209873944} Y_6 \\
&+ \frac{24706724315212555 + 23536035499434854i\sqrt{2}}{882959697} Y_7 \\
&+ \frac{1344530700471761 + 7054273282673287i\sqrt{2}}{92943126} Y_8 \\
&+ \frac{-6254424154733655563 + 2817810866364897191 8i\sqrt{2}}{3145055128201595952} Y_9
\end{aligned}
$$

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- We started with four pairwise products of the form $W_i W_j$ with previously calculated equations in terms of a basis for $H^0(\widehat{FPP}, 2K)$.

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- We started with four pairwise products of the form $W_i W_j$ with previously calculated equations in terms of a basis for $H^0(\widehat{FPP}, 2K)$.
- Each W_i individually is one of the basis sections of $H^0(9\widehat{FPP}, K)$, so we next separated the products $W_i W_j$ into their individual components.

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- We started with four pairwise products of the form $W_i W_j$ with previously calculated equations in terms of a basis for $H^0(\widehat{FPP}, 2K)$.
- Each W_i individually is one of the basis sections of $H^0(9\widehat{FPP}, K)$, so we next separated the products $W_i W_j$ into their individual components.
- We then used the nonreduced linear cut to construct the 2-to-1 cover $18FPP \to 9\widehat{FPP}$. To start, there is a section $s^2 \in H^0(\widehat{FPP}, 2K)$ such that $s^2 =$ *equation for nonreduced cut*.

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- We then pull back s^2 to $H^0(18FPP, 2K)$, and since the pullback is a perfect square, we get a section $s \in H^0(18FPP, K)$.

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- We then pull back s^2 to $H^0(18FPP, 2K)$, and since the pullback is a perfect square, we get a section $s \in H^0(18FPP, K)$.
- We then used this s along with the eight W_i to construct a basis of the 17-dimensional space $H^0(18FPP, K)$; call this basis B.

Results: Extending the Group Action

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- This allows us to get equations for the FPPs by looking for relations between sections of $H^0(18FPP, 2K)$ that are invariant on a subgroup of $C_3 \times C_3 \times S_3$.
- However, the coefficients we got when we solved for the S_3 action were incredibly ugly, so we want to find a better basis than B before looking for equations (this will make the equations we find nicer).

What's Left

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- We then search for degree 3 polynomial relations between the elements of B_i, which will give us the desired equations.

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