High-School Partitions

REU 2020 PROJECT

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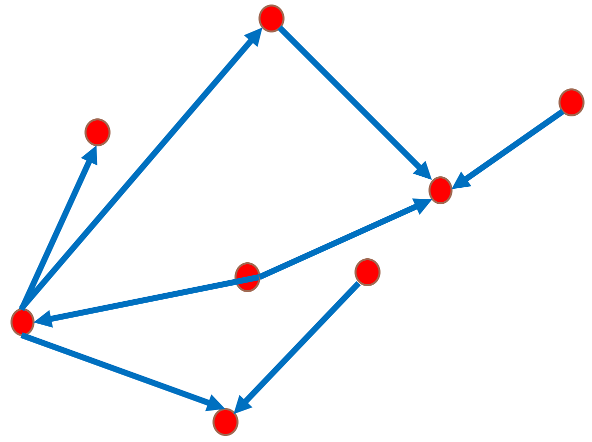
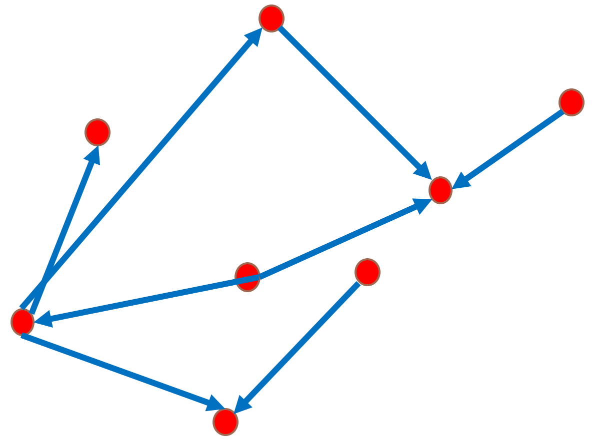
Legal Partitions

- $G = (V, E)$ – digraph
- V – set of students
- E – preferences:
  - $u \to v \in E \quad \Longrightarrow \quad u$ wants to be with v



Legal Partitions



- $G = (V, E)$ – digraph

Definition.
Let $V = V_1 \cup \cdots \cup V_k$ be a partition of V into k nonempty parts.
The partition is called <u>legal</u> if

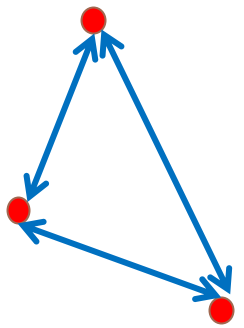
$$(\forall V_i)(\forall v \in V_i)\colon \mathrm{outdeg}_{V_i}(v) \geq 1$$

- So, we partition to k classes such that each student has at least 1 friend in each class

Existence of Legal Partitions

Theorem [Thomassen 1983].
Let $G = (V, E)$ be a graph such that each vertex has outdegree at least 3. Then, there exists a legal partition $V = V_1 \cup V_2$.

- Theorem is tight:



Thomassen's Theorem

Proof [Overview].

- Constructive proof
- WLOG assume the outdegree of each vertex is <u>exactly</u> 3
- Two steps
  1. Find two disjoint cylces (interesting part)
  2. Extend the two cycles to the desired partition (easy part)

Extending Disjoint Cycles to a Legal Partition

Let C_1, C_2 denote the disjoint cycles obtained in Step 1.

Consider a maximal legal extension $C_1 \subseteq V_1, C_2 \subseteq V_2$.

We claim that $V = V_1 \cup V_2$ (and hence a legal partition)

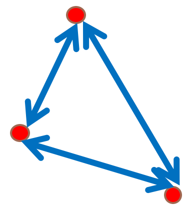
Indeed, if $\exists v \notin V_1 \cup V_2$ then consider any maximal path starting at v

If this path reaches V_1 or V_2 at some point then add the corresponding prefix to V_1 or V_2 accordingly

Else, add the entire path to V_1 or V_2 arbitrarily

Q1: How Many Legal Partitions Are There?

Question 1.
Let $t(d,n)$ denote the minimum number of non-trivial legal partitions of a digraph with n vertices and minimum outdegree $\geq d$. Provide upper and lower bounds on $t(d,n)$.



- $t(2,n) = 0$
- $t(3,n) \geq 1$

\\\\Thomassen Theorem

- $\lim_{n\to\infty} t(3,n) = \infty$???
- How about $t(d,n)$ for larger d's?

Q2: Can Any Pair of Vertices Be Separated Legally?

Question 2.
Let $G = (V, E)$ be a graph so that each vertex has outdegree at least 3 and let $u, v \in V$ be distinct. Does there exist a legal partition $V = V_1 \cup V_2$ such that $u \in \mathrm{V}_1$, $v \in V_2$???

- An affirmitive answer implies $t(3, n) \geq \lfloor \log n \rfloor$
- How about when we assume a larger outdegree?
  - Constant?
  - Some function of n?