Simplifying explicit equations of fake projective planes

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Origin of Fake Projective Planes

Definition

The complex projective plane $\mathbb{C}P^2$ is the collection of 1-dimensional linear subspaces of $\mathbb{C}^3$.

- $\mathbb{C}P^2$ is a canonical example of a complex surface (with 4 real dimensions).
- A question by Severi asked are there any complex surfaces that are topologically equivalent to $\mathbb{C}P^2$ but not biholomorphic to it.
- The answer is negative by a corollary of the famous Calabi-Yau Theorem in 1977:

Corollary in [Yau77]

Every complex surface that is homotopic to $\mathbb{C}P^2$ is biholomorphic to it.

Origin of Fake Projective Planes

There are some topological properties on $\mathbb{C}P^2$ that's more subtle than homotopy equivalence:

- For a complex surface, its k-th Betti number is intuitively a count of how many k-dimensional holes the surface has.
- The Euler characteristics of the surface is the alternating sum of its Betti numbers.
- For $\mathbb{C}P^2$, its Betti numbers are $1, 0, 1, 0, 1$ for dimensions $n = 0 - 4$ and its Euler characteristics is 3.

Mumford considered a weaker question in [Mum79]:

Question:

Does there exist a complex surface with the same Betti numbers as $\mathbb{C}P^2$ but not biholomorphic to it?

- In the same paper, Mumford constructed the first example of such surface.
- These surfaces are now known as fake projective planes.

Properties of Fake Projective Planes

Question:

Why should one care about fake projective planes?

- By Chow's Theorem, it is known that all fake projective planes are algebraic surfaces, meaning that they can be written as the zero locus of a collection of complex polynomial equations [Cho49].
- Fake projective planes are canonical examples of "algebraic surfaces of general type" with the smallest Euler characteristics.
- Studying fake projective planes would help us understand the geometric classifications of these surfaces of general type.

Classification of Fake Projective Planes

There's a geometric description of what all fake projective planes are:

Definition:

Let $\mathbb{D}^2 \subset \mathbb{C}^2$ denote the complex 2-ball:

$$\mathbb{D}^2 := \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 < 1\}$$

A corollary of the Calabi-Yau Theorem shows that

Corollary:

Every fake projective plane can be realized as the quotient of $\mathbb{D}^2$ with a torsion-free cocompact discrete subgroup of $PU(2,1)$.

Classification of Fake Projective Planes

By the works of Cartwright and Steger in [CS10] and Prasad and Yeung in [PY10], all possible quotients here are determined, and it is known that there are exactly 100 fake projective planes up to biholomorphism.

Computational Perspective

Recall earlier that all fake projective planes are algebraic,

Question:

Suppose we are given a fake projective plane M, can we find explicit polynomials that constructs M? How complicated are the polynomials?

- In 2018, Borisov and Keum constructed explicit equations in [BK20] for certain examples of fake projective planes.
- In general, the equations of fake projective planes are long with coefficients of thousands of digits.
- Not all fake projective planes have explicit equations constructed yet!
- The goal of our REU is to simplify some known equations of fake projective planes and find explicit equations for other fake projective planes.

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