Computing explicit equations of fake projective planes

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Outline

Fake Projective Plane

Definition:

The complex projective n-space $\mathbb{C}P^n$ is the set of 1-dimensional linear subspaces of $\mathbb{C}^{n+1}$.

Definition:

A fake projective plane (FPP) is a complex surface with the same Betti numbers as $\mathbb{C}P^2$ but not biholomorphic to it.

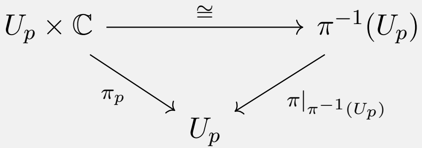
- By Chow's Theorem, it is known that all FPPs are algebraic surfaces [Cho49].
- By the works of Cartwright and Steger in [CS10] and Prasad and Yeung in [PY10], it is known that there are exactly 100 fake projective planes up to biholomorphism.
- However, the question of constructing explicit equations for each FPP is largely unsolved. For FPPs with equations, they tend to be quite complicated.

Line Bundles

Definition:

A (complex, holomorphic) line bundle $\pi : L \to M$ is a holomorphic map such that:

- For all $p \in M$, $\pi^{-1}(p)$ is biholomorphic to $\mathbb{C}$.
- (Local Triviality): For all $p \in M$, there exists some neighborhood U_p such that the following diagram commutes and are all holomorphic maps:



A global section of L is a map $s : M \to L$ such that $\pi \circ s = 1_M$. $H^0(M, L)$ is the $\mathbb{C}$-vector space of all global sections.

Examples and the Picard Group

Examples:

- The line bundle $\pi : \mathbb{C} \times M \to M$ is an example of a globally trivial line bundle.
- The tangent bundle of $\mathbb{C}P^1$ is an example of a non-trivial line bundle.

Definition:

The isomorphism classes of line bundles $L \to M$ form an abelian group known as the "Picard Group" $\mathrm{Pic}(M)$ via tensor product.

How to Construct Maps to $\mathbb{C}P^n$

Let X be a closed complex manifold and $\pi : L \to M$ be a line bundle. Suppose for every $p \in M$, there exists some section s such that $s(p) \neq 0$.

1. Since M is compact, $\dim_{\mathbb{C}} H^0(M, L) < \infty$, we can find a basis $s_0, ..., s_n$ of $H^0(M, L)$.
2. We construct a map $\Phi : M \to \mathbb{C}P^n$ as

   $$\Phi(p) = [s_0(p) : s_1(p) : ... : s_n(p)]$$

   $s_0, ..., s_n$ can't be all 0 at the same point due to our assumption.
3. Technically, each $s_i(p)$ is an element of $\pi^{-1}(p)$, but it is without loss a complex number via the isomorphism $\pi^{-1}(p) \cong \mathbb{C}$. This is well-defined by the equivalence of the projective coordinates.

Outline

The Fake Projective Plane of Interest

We are interested in the fake projective plane $X = (a = 7, p = 2, \emptyset, D_3 2_7)$ in the Cartwright-Steger Classification [CS10].

- **Fact:** There exists a line bundle H over X such that the canonical line bundle K is isomorphic to $3H$.
- Cartwright and Steger realized X as a quotient $\mathbb{B}^2/\Gamma_1$ in [CS10], where

$$\mathbb{B}^2 := \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 < 1\}$$

  It is also shown that

$$\operatorname{Pic}(X) \cong \mathbb{Z}H \oplus (\mathbb{Z}/2)^4$$

- There exists a unique non-zero torsion element $D \in \operatorname{Pic}(X)$ that's fixed by automorphisms of X.
- In [BK20], Borisov and Keum produced an embedding of X into $\mathbb{C}P^9$ using 10 global sections of $H^0(X, 6H)$.

Outline

Main Result I

By the Riemann-Roch Theorem and the Kodaira Vanishing Theorem, one can find that, for any torsion $T \in \mathrm{Pic}(X)$ and $n \geq 4$,

$$\dim_{\mathbb{C}} H^0(X, nH + T) = \frac{(n-1) \cdot (n-2)}{2}$$

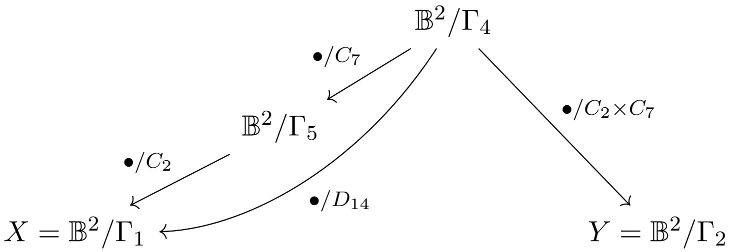
Result I:

- By considering the basis of 6 global sections on $H^0(X, 5H + D)$, we were able to produce an embedding of X into $\mathbb{C}P^5$.
- Interestingly, when we tried to do the same procedure with the 6 global sections of $H^0(X, 5H)$, the image of X is singular.

Main Result II

Result II:

We found an explicit embedding for a new pair of fake projective planes $Y = (a = 7, p = 2, \emptyset, D_3X_7)$ in $\mathbb{C}P^9$ closely related to X.



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References I

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Background in Algebraic/Complex Geometry

The Fake Projective Plane of Interest

Main Results

Lev A. Borisov and JongHae Keum.
Explicit equations of a fake projective plane.
Duke Mathematical Journal, 169(6), apr 2020.

Wei-Liang Chow.
On compact complex analytic varieties.
American Journal of Mathematics, 71(4):893–914, 1949.

Donald I. Cartwright and Tim Steger.
Enumeration of the 50 fake projective planes.
Comptes Rendus Mathematique, 348(1):11–13, 2010.

Gopal Prasad and Sai-Kee Yeung.
Addendum to "fake projective planes" invent.nbsp;math. 168, 321–370 (2007).
Inventiones mathematicae, 182(1):213–227, 2010.