Grid Multi-Robot Path Planning with the Corner Following Constraint: Tighter Bounds and Intractability

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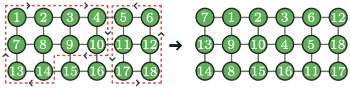
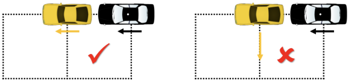
Multi-Robot Path Planning (MRPP) Problems

Grid MRPP: Suppose you have a grid graph $G(V, E)$ and n robots with start configuration $S = \{s_1, \ldots, s_n\} \subseteq V$ and goal configuration $G = \{g_1, \ldots, g_n\} \subseteq V$. We seek to route the robots along collision-free paths P_i efficiently. In particular, we seek to minimize the overall makespan where motion is uniform without any meet or head-on collisions.

MRPP with CFC: Additionally, we may have a corner following constraint.



(a) Single time step move [1]

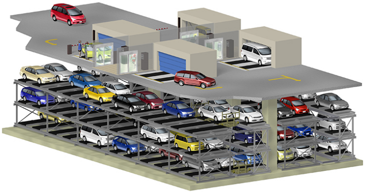


(b) Corner Following Constraint [2]

Motivation

(a) Ocado Systems [a]

[a] "Inside a Warehouse Where Thousands Of Robots Pack Groceries." Tech Insider.



(b) Automated Garages [a]

[a] "Automated Multistory Parking Facilities." Trends in Japan.

Known Results

Rubik Tables: an $m \times n$ grid can be arbitrarily permuted in three phases, permuting m rows, n columns, and then m rows again [1].

Grid MRPP: sub-1.5 time-optimal algorithms via Rubik Tables [3].

MRPP with CFC:

	$\Theta(m_1 m_2)$ Escorts	$\Theta(m_1 + m_2)$ Escorts	$\Theta(1)$ Escorts
Upper Bound	$\mathrm{O}(\mathrm{m}_1 + m_2)$	$O(m_1 m_2)$*	-*
Lower Bound	h.p. $\Omega(m_1 + m_2)$	h.p. $\Omega(m_1 + m_2)$	exp. $\Omega(m_1 m_2)$

Note: escorts are the empty spaces in our MRPP instance.

New Results

Theorem (Single Escort MRPP with CFC Makespan)

A given MRPP with CFC instance on an $m_1 \times m_2$ grid for $m_1, m_2 \geq 2$ containing a single escort can be solved in $O(m_1 m_2)$ time.

This matches the $\Omega(m_1 m_2)$ expected lower bound.

Theorem (k Escorts MRPP with CFC Makespan)

A given MRPP with CFC instance on an $m_1 \times m_2$ grid for $m_1, m_2 \geq 2$ containing $k \leq \min(m_1, m_2)$ escorts can be solved in $O(m_1 m_2 / k)$ time, and for $k > min(m_1, m_2)$, it can be solved in $O(\max(m_1, m_2))$ time.

These match the $\Omega(m_1 m_2 / k)$ expected lower bound and $\Omega(\max(m_1, m_2))$ high probability lower bound respectively.

Single Escort Upper Bound

Overall Strategy: Rubik Tables

How can we efficiently simulate row and column permutations?

Single Escort Upper Bound

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Lemma (Linear $3 \times m$ Sort)

Given a single escort, a $3 \times m$ grid an be arbitrarily permuted except possibly with two adjacent tiles swapped, which can be arbitrarily chosen, in $O(m)$ time.

Single Escort Upper Bound

Overall Strategy: Rubik Tables
How can we efficiently simulate row and column permutations?

Lemma (Linear $3 \times m$ Sort)

Given a single escort, a $3 \times m$ grid an be arbitrarily permuted except possibly with two adjacent tiles swapped, which can be arbitrarily chosen, in $O(m)$ time.

Lemma (Linear $2 \times m$ Sort)

Given a single escort, a $2 \times m$ grid an be arbitrarily permuted except possibly with two adjacent tiles swapped, which can be arbitrarily chosen, in $O(m)$ time.

Single Escort Upper Bound

Overall Strategy: Rubik Tables
How can we efficiently simulate row and column permutations?

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Given a single escort, a $2 \times m$ grid an be arbitrarily permuted except possibly with two adjacent tiles swapped, which can be arbitrarily chosen, in $O(m)$ time.

Corollary (Linear $2 \times m$ Half-Sort)

Given a single escort, a $2 \times m$ grid can be permuted to fill one of its rows arbitrarily in $O(m)$ time.

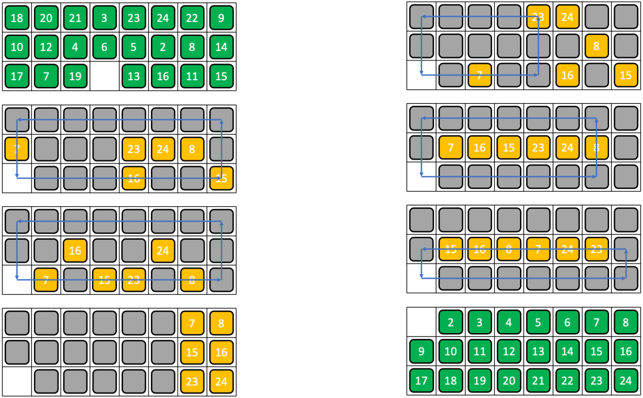
Linear $3 \times m$ Sort

How can we permute the robots while leveraging batched movement with just a single escort?

Linear 3 × m Sort

How can we permute the robots while leveraging batched movement with just a single escort?

1. We solve a subgrid, ignoring the other elements, giving us "room" to arrange the chosen elements.
2. We use a "highway" to move elements in batches into the right order.



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MRPP with CFC Related Decision Problems

$\mathbf{MRPP_{CFC}}$: The corresponding decision problem for general MRPP with CFC, which asks whether there exists a solution routing with a makespan bounded by a given number k.

- Motivation: automated garages, warehouse automation, ...

SnPUZ: Single Escort $MRPP_{CFC}$, or Synchronous nPUZ (refering to the $(n^2 - 1)$-puzzle, which is known to be NP-hard [4], [5])

- Motivation: making full use of automated garages in locations where space is a premium

BSnPUZ: binary SnPUZ, i.e. tiles are black or white

- Motivation: rearranging quickly when exact object placement isn't important, e.g. AG pickup interval

PSnPUZ: partial SnPUZ, i.e. tiles are unique except for some number of indistinguishable white tiles

- Motivation: partially sorting AG prioritizing urgent (time or importance) pickup times

New Intractability Results*

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The above intractability holds if we require exactly $\lfloor n^\epsilon \rfloor$ escorts/white tiles for a fixed constant $0 < \epsilon < 2$. In addition, they all belong to NP and thus are NP-complete.

Acknowledgements

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