GENERALIZING THE FKG INEQUALITY

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INTRODUCTION

- Consider a finite distributive lattice L equipped with a positive log-supermodular probability measure function,

$$\mu(x \vee y)\mu(x \wedge y) \geq \mu(x)\mu(y)$$

- By the FKG inequality, for any two positive monotone (increasing or decreasing) functions f, g on L,

$$Cov(f, g) = \mathcal{E}(fg) - \mathcal{E}(f)\mathcal{E}(g) \geq 0$$

- In other words, positive monotone functions on a distributive lattice are **positively correlated**.

A NATURAL EXTENSION

- It is conjectured that the FKG inequality holds in a broader setting. Define the multilinear functional $E_n(f_1, \dots, f_n)$:
  - Decompose a permutation σ in the symmetric group S_n as a product of disjoint cycles: $\sigma = (i_1, \dots, i_p)(j_1, \dots, j_q) \dots$ (1)
  - For σ as in (1), let C_σ denote the number of cycles in σ and define:

$$E_\sigma(f_1, \dots, f_n) = \mathcal{E}\left(f_{i_1} \dots f_{i_p}\right) \mathcal{E}\left(f_{j_1} \dots f_{j_q}\right) \dots \qquad (2)$$

  - From (1) and (2), we get E_n:

$$E_n(f_1, \dots, f_n) = \sum_{\sigma \in S_n} (-1)^{C_\sigma - 1} E_\sigma(f_1, \dots, f_n)$$

AN IMPORTANT CONJECTURE

- A distributive lattice L with log-supermodular positive probability measure μ will be referred to as an *FKG poset.*
- The following conjecture is from the paper *"On the extension of the FKG inequality to n functions"* by Siddhartha Sahi and Elliott H Lieb.
- $\boldsymbol{Conjecture}$: $If\ f_1, \ldots, f_n\ are\ positive\ monotone\ functions\ on\ an\ FKG$ $poset\ then$ $$E_n(f_1, \ldots, f_n) \geq 0$$

  1. For $n = 1, E_1(f_1) = \mathcal{E}(f_1) \geq 0$ since f_1 is a positive function
  2. For $n = 2, E_2(f_1, f_2) = \mathcal{E}(f_1 f_2) - \mathcal{E}(f_1)\mathcal{E}(f_2) \geq 0$ by the *FKG inequality*.
  3. For $n = 3, E_3(f, g, h) =$ $$2\mathcal{E}(fgh) + \mathcal{E}(f)\mathcal{E}(g)\mathcal{E}(h) - \mathcal{E}(f)\mathcal{E}(gh) - \mathcal{E}(g)\mathcal{E}(fh) - \mathcal{E}(h)\mathcal{E}(fg)$$

     (has not been shown to be non-negative yet)

PROVING THE CONJECTURE

- The first step is to consider monotone Boolean functions,

  $f\colon \{0,1\}^n \rightarrow \{0,1\}$ and $\boldsymbol{x} \leq \boldsymbol{y} \Rightarrow f(\boldsymbol{x}) \leq f(\boldsymbol{y})$ (monotone increasing)

- The number of monotone Boolean functions (MBFs) in n variables rapidly increases as n increases. The number of MBFs in n variables is given by the *Dedekind numbers.*
- The exact values are known for $0 \leq n \leq 8$,
  2, 3, 6, 20, 168, 7581, 7828354, 2414682040998, 56130437228687557907788
- The MBFs prove to be an important class of functions; they form the extremities of a convex set containing functions of the form

  $f\colon \{0,1\}^n \rightarrow [0,1]$ and $\boldsymbol{x} \leq \boldsymbol{y} \Rightarrow f(\boldsymbol{x}) \leq f(\boldsymbol{y})$ (monotone increasing)

PREVIOUS WORK

- The multilinear functionals E_n possess an important algebraic property:

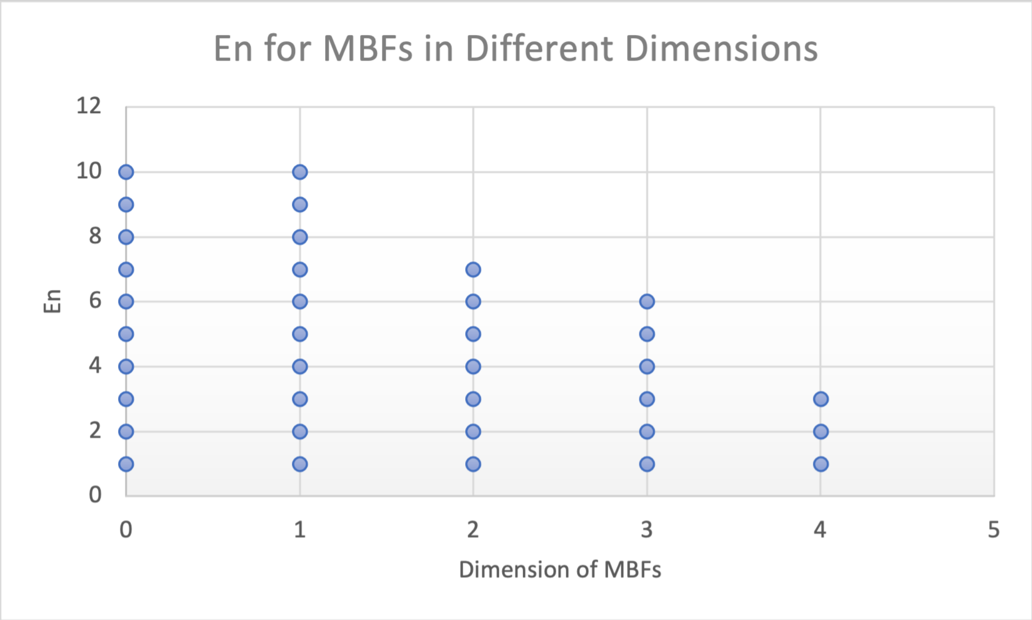
$$E_n(f_1, \ldots, f_{n-1}, f) = e_1 + \ldots + e_{n-1} - e_n \text{ , where}$$

$$e_i = \begin{cases} E_{n-1}(f_1, \ldots, f_i f, \ldots, f_{n-1}) & \text{if } 1 \leq i \leq n-1 \\ E_{n-1}(f_1, \ldots, f_{n-1})\mathcal{E}(f) & \text{if } i = n \end{cases}$$

- These relations can be used to compute $E_1, E_2, \ldots$ recursively for MBFs in a fixed number of dimensions.
- Since a Boolean lattice is isomorphic to the **power set** of the atoms of the Boolean lattice, the MBFs can also be recursively generated.
- The algebraic property mentioned above endows E_n with a **hierarchical property.**

PREVIOUS WORK

- The graph below shows the number of variables of the MBFs for which $E_n \geq 0$ has been verified.



FUTURE WORK

- The *Conjecture on* E_n by itself involves a massive class of functions. It makes sense to check the validity on individual classes of functions by sampling functions and testing the hypothesis. An example of this is the validity of $\boldsymbol{F_n}$.
- If E_n holds true, albeit with some conditions imposed (possibly on the class of functions), it may hold applications in discovering new properties of:
  - **Log-supermodular functions**
  - **Bernstein polynomials**
  - **Log convex sequences**
  - **Tutte polynomials** of a geometric lattice

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