Extension of the FKG Inequality

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Introduction

We consider a finite distributive lattice L equipped with a special probability measure function,

$$\mu(x \vee y)\mu(x \wedge y) \geq \mu(x)\mu(y)$$

By the FKG Inequality, for any two positive monotone functions f,g on L,

$$E(xy) - E(x)E(y) \geq 0$$

Essentially, positive monotone functions on a distributive lattice are positively correlated.

Example

- Let $G = (V, E)$ be a random graph on V obtained by picking every edge, independently, with probability p.
- Let P denote the property that the graph is Planar and H denote the property that the graph is Hamiltonian.
- P is a monotonically decreasing property since every graph G' on the same vertices which is a sub-graph of G is also planar.
- H is a monotonically increasing property since every graph G on the same vertices which contains G as a sub-graph is also Hamiltonian.
- The set of edges can be viewed as a Boolean lattice and taking μ to be the product measure, we can apply the FKG Inequality to get:

$$Pr(P \wedge H) \leq Pr(P)Pr(H) \Leftrightarrow Pr(P|H) \leq Pr(P)$$

Expression for E_n

The multi-linear functional E_n can be viewed as an extension for the FKG Inequality. E_n is defined as below:

1. Decompose a permutation σ in the symmetric group S_n as a product of disjoint cycles:
$$\sigma = (i_1, ..., i_p)(j_1, ..., j_q)...$$
2. For σ as above, let C_σ denote the number of cycles in σ and define:
$$E_\sigma(f_1, ..., f_n) = E(f_{i_1}, ..., f_{i_p})E(f_{j_1}, ..., f_{j_q})...$$
3. Combining the above two expressions, we get E_n:
$$E_n(f_1, ..., f_n) = \sum_{\sigma \in S_n} (-1)^{C_\sigma - 1} E_\sigma(f_1, ..., f_n)$$

Conjecture

Siddhartha Sahi conjectured that the multi-linear functional E_n is non-negative for positive monotone functions on FKG posets. For $n = 1, 2$ E_n is non-negative.

1. $n = 1$:
$$E_1(f) = E(f) \geq 0$$
by non-negativity of f.
2. $n = 2$:
$$E_2(f, g) = E(fg) - E(f)E(g) \geq 0$$
by the FKG Inequality.
3. $n = 3$:
$$E_3(f, g, h) = 2E(fgh) + E(f)E(g)E(h) - E(f)E(gh) - E(g)E(fh) - E(h)E(fg)$$
which is conjectured to be non-negative.

Expression for F_n

F_n is defined almost identically to E_n except instead of point-wise multiplication of functions we take point-wise minima of functions.

We also restrict the range of the monotone Boolean functions to the closed interval [0,1]. For example the expression for F_3 is:

$$F_3(f,g,h) = 2E(f*g*h) + E(f)E(g)E(h) - E(f)E(g*h) - E(g)E(f*h) - E(h)E(f*g)$$

where $f*g$ is the point-wise minima taken across all points of the Boolean Lattice.

Showing that $F_n \geq 0$ is a stronger version of the general $E_n \geq 0$. When we consider characteristic functions i.e., $\{0,1\}$ valued functions, F_n reduces to E_n. Characteristic functions over sets are some of the simplest examples of monotonically increasing functions and showing that E_n is non-negative over them would be an important result.

Monotone Boolean Functions

A Boolean function takes Boolean variables as input; the dimension of the function is given by the number of Boolean variables it is a function of.

A one dimensional Boolean function can be treated as a point in $[0,1]^2$. Its value at 0 gives one coordinate and its value at 1 gives the other coordinate.

A two dimensional Boolean function can be treated as a point in $[0,1]^4$. Its values at $\{00\}, \{01\}, \{10\}, \{11\}$ give the four coordinates.

Since these functions are also monotonically increasing they must also satisfy the condition that:

$$x \leq y \implies f(x) \leq f(y)$$

Geometric Perspective

The set of one-dimensional monotone Boolean functions MBF_1 has a simple geometric visualization.

$$MBF_1 = \{(x, y) \in [0, 1]^2 | x \leq y\}$$

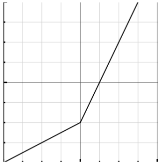
The set of two-dimensional monotone Boolean functions MBF_2 has a slightly more complex geometric visualization.

$$MBF_2 = \{(x_1, x_2, x_3, x_4) \in [0, 1]^4 | x_1 \leq \min(x_2, x_3), x_4 \geq \max(x_2, x_3)\}$$

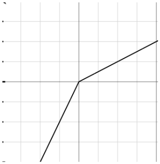
F_n can be understood as a function over k-space to $[0, 1]$, where $k = n * 2^d$ and d is the dimension of each MBF.

Quasi-concavity

Let S be a convex set. A function $f : S \to \mathbb{R}$ is quasi-concave if for each $a, b \in S$, $f(a + tb) \geq \min(f(a), f(b))$ $\forall t \in [0, 1]$.
The analog of F_n in a 2-dimensional plane is a piece-wise linear function. In reality it is a piece-wise planar graph; F_n is almost linear except at the points at which the *min* function changes its nature.



Quasi-concave



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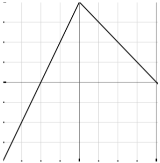

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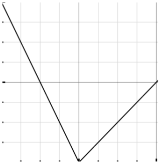

Not Quasi-concave

Quasi-concavity

If a function is quasi-concave on its domain, then it is easy to see that the function can only take its minima at the extreme points of the convex set (points which do not lie along any line contained in the convex set).

Thus verifying non-negativity of F_n across its domain would reduce to checking non-negativity at its extreme points.

An easy way to check quasi-concavity is to fix $n-1$ functions in F_n and vary the n^{th} function. In this case, the convex set S is the set of all d-dimensional MBFs.

Counter-example for Quasi-concavity

We began testing the quasi-concave property for F_3 but ended up finding a counter-example (Here F_3 is over one-dimensional MBFs).

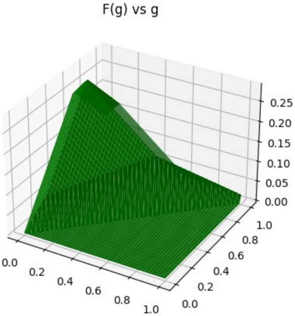

Figure: F3 as a function of g (f0=[0.6,0.6], f1=[0.1,1])

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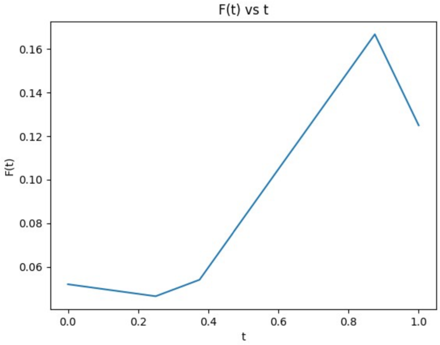

Figure: F3 as a function of t (f0=[0.6,0.6], f1=[0.1,1], g0=[0,1], g1=[0.9,0.9])

Searching for Minima

F_3 fails the quasi-concavity test which means that it may have minima in its non-extreme points. These putative minima may be of two kinds:

1. Zero minima: Minima at which F_3 is zero
2. Non-zero minima: Minima at which F_3 is positive

There are numerous examples of zero minima; the simplest one is two of the functions being identically zero while the third one can be any function belonging to the convex set. Then F_3 is zero and the point is a non extreme point.

Siddhartha and I began searching for an algorithm that could generate these non-extreme non-zero minima.

Algorithm

This is the algorithm which given a random initial point, tries to decrease F_3 along connected points in the domain until it either terminates with a zero minima or a non-zero minima. This algorithm runs for F_3 for d-dimensional MBFs. Let $S \subset [0,1]^k$ denote the space of n many d-dimensional MBFs ($k = n * 2^d$).

1. Begin with a random point p in S. p is a k-tuple.
2. Repeat until p does not change over k successive iterations.
    1. For each co-ordinate of p:
        1. Compute $F_3(p1_{up})$ and $F_3(p1_{down})$
        2. $p = p1_{up}$ if $F_3(p1_{up}) \leq F_3(p1_{down})$ and $p = p1_{down}$ otherwise

The p obtained from the previous steps is used for the next part of the algorithm on the next slide.

Algorithm

1. Repeat until p does not change over n successive iterations.
   1. For each of the n functions:
      1. Partition the function's co-ordinates, $\{1, ..., 2^d\}$, into subsets based on whether p has the same value for indices of the subset.
      2. For each set in the partition:
         1. Compute $F_3(p2_{up})$ and $F_3(p2_{down})$
         2. $p = p2_{up}$ if $F_3(p2_{up}) \leq F_3(p2_{down})$ and $p = p2_{down}$ otherwise

After running the above algorithm a large number of times for $d = 1$ and 2 the algorithm always terminated at a zero minima. The mechanism used by the algorithm to reduce points to zero minima motivated me to search for a proof for showing that F_3 over one-dimensional MBFs has no non-zero minima.

Understanding the finer workings of the algorithm might provide a clue for showing that F_3 over all MBFs has no non-zero minima (no restriction on dimension of MBFs).

Sketch of Proof

Proposition: F_3 over 1-dimensional MBFs has no non-zero minima.

Sketch of Proof:

- We prove the proposition by contradiction. Suppose F_3 does have a non-zero minima, say $(f_1, f_2, g_1, g_2, h_1, h_2)$.
- Since this point belongs to our convex set S we have the following restrictions: $0 \leq f_i, g_i, h_i \leq 1$ and $f_1 \leq f_2, g_1 \leq g_2, h_1 \leq h_2)$.
- Despite all these restrictions, there is still freedom in the linear ordering of the sets $\{f_1, g_1, h_1\}$ and $\{f_2, g_2, h_2\}$.
- By the symmetric nature of F_3 ($F_3(f, g, h) = F_3(g, f, h) = ..$), w.l.o.g. we can also assume $f_1 \leq g_1 \leq h_1$
- This leaves us with six cases based on the ordering of $\{f_2, g_2, h_2\}$

Sketch of Proof

- The co-ordinate wise orderings are important since they give an explicit formula for F_3. By looking at the sign of the coefficients of these co-ordinates we can claim that these co-ordinates must in fact be 0 or 1.
- Reductions of this type result in finally arriving at a contradiction: Step by step we impose that certain co-ordinates take specific values like 0 or 1 (the violation of which would contradict minimality of the point) and finally show that these steps lead to a zero minima, thus contradicting our initial assumption that the point was a non-zero minima.
- By considering these 6 broad cases, this argument holds true in general for any point belonging to S.

Conclusion

- Studying the nature of the minima of F_n is an important step in attempting to show that F_n is non-negative.
- I have successfully showed that F_3 over one-dimensional MBFs has no non-zero minima.
- Studying how the algorithm eventually reaches a zero minima for F_3 over any MBF will provide an insight into a potential argument for a formal proof.
- A correlation inequality in 3 monotone functions has applications in probability theory, combinatorics, stochastic processes and statistical mechanics.
- It would be exciting to see a stronger version of E_3 being applied in areas like uniform random spanning tree measures, symmetric exclusion processes, random cluster models (with q < 1), balanced and Rayleigh matroids.

References

- “On the extension of the FKG inequality to n functions” by Siddhartha Sahi and Elliott H Lieb
- “Higher Correlation Inequalities” by Siddhartha Sahi
- ”Combinatorial applications of an inequality from statistical mechanics” by P.D. Seymour and D.J.A. Welsh
- ”Matroid Theory” by James G. Oxley
- ”Balanced Matroids” by Tomas Feder and Milena Mihail
- ”Negative Correlation in Graphs and Matroids” by Charles Semple and Dominic Welsh
- ”Correlation bounds for fields and matroids” by June Huh, Benjamin Schroter and Botong Wang

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