Extension of the FKG Inequality

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July 22, 2022
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We consider a finite distributive lattice $L$ equipped with a special probability measure function,

$$
\mu(x \lor y)\mu(x \land y) \geq \mu(x)\mu(y)
$$

By the FKG Inequality, for any two positive monotone functions $f, g$ on $L$,

$$
E(xy) - E(x)E(y) \geq 0
$$

Essentially, positive monotone functions on a distributive lattice are positively correlated.
Let $G = (V, E)$ be a random graph on $V$ obtained by picking every edge, independently, with probability $p$.

Let $P$ denote the property that the graph is Planar and $H$ denote the property that the graph is Hamiltonian.

$P$ is a monotonically decreasing property since every graph $G'$ on the same vertices which is a sub-graph of $G$ is also planar.

$H$ is a monotonically increasing property since every graph $G$ on the same vertices which contains $G$ as a sub-graph is also Hamiltonian.

The set of edges can be viewed as a Boolean lattice and taking $\mu$ to be the product measure, we can apply the FKG Inequality to get:

$$Pr(P \land H) \leq Pr(P)Pr(H) \iff Pr(P|H) \leq Pr(P)$$
Expression for $E_n$

The multi-linear functional $E_n$ can be viewed as an extension for the FKG Inequality. $E_n$ is defined as below:

1. Decompose a permutation $\sigma$ in the symmetric group $S_n$ as a product of disjoint cycles:

$$\sigma = (i_1, ..., i_p)(j_1, ..., j_q)....$$

2. For $\sigma$ as above, let $C_\sigma$ denote the number of cycles in $\sigma$ and define:

$$E_\sigma(f_1, ..., f_n) = E(f_{i_1}, ..., f_{i_p})E(f_{j_1}, ..., f_{j_q})....$$

3. Combining the above two expressions, we get $E_n$:

$$E_n(f_1, ..., f_n) = \sum_{\sigma \in S_n} (-1)^{C_\sigma - 1} E_\sigma(f_1, ..., f_n)$$
Conjecture

Siddhartha Sahi conjectured that the multi-linear functional $E_n$ is non-negative for positive monotone functions on FKG posets. For $n = 1, 2$, $E_n$ is non-negative.

1. $n = 1$: 

$$E_1(f) = E(f) \geq 0$$ 

by non-negativity of $f$.

2. $n = 2$: 

$$E_2(f, g) = E(fg) - E(f)E(g) \geq 0$$ 

by the FKG Inequality.

3. $n = 3$: 

$$E_3(f, g, h) = 2E(fgh) + E(f)E(g)E(h) - E(f)E(gh) - E(g)E(fh) - E(h)E(fg)$$ 

which is conjectured to be non-negative.
Expression for $F_n$

$F_n$ is defined almost identically to $E_n$ except instead of point-wise multiplication of functions we take point-wise minima of functions.

We also restrict the range of the monotone Boolean functions to the closed interval $[0,1]$. For example the expression for $F_3$ is:

$$F_3(f, g, h) = 2E(f \ast g \ast h) + E(f)E(g)E(h) - E(f)E(g \ast h) - E(g)E(f \ast h) - E(h)E(f \ast g)$$

where $f \ast g$ is the point-wise minima taken across all points of the Boolean Lattice.

Showing that $F_n \geq 0$ is a stronger version of the general $E_n \geq 0$. When we consider characteristic functions i.e., $\{0, 1\}$ valued functions, $F_n$ reduces to $E_n$. Characteristic functions over sets are some of the simplest examples of monotonically increasing functions and showing that $E_n$ is non-negative over them would be an important result.
Monotone Boolean Functions

A Boolean function takes Boolean variables as input; the dimension of the function is given by the number of Boolean variables it is a function of.

A one dimensional Boolean function can be treated as a point in [0, 1]. Its value at 0 gives one coordinate and its value at 1 gives the other coordinate.

A two dimensional Boolean function can be treated as a point in [0, 1]. Its values at \{00\}, \{01\}, \{10\}, \{11\} give the four coordinates.

Since these functions are also monotonically increasing they must also satisfy the condition that:

\[ x \leq y \implies f(x) \leq f(y) \]
The set of one-dimensional monotone Boolean functions $MBF_1$ has a simple geometric visualization.

$$MBF_1 = \{(x, y) \in [0, 1]^2 | x \leq y\}$$

The set of two-dimensional monotone Boolean functions $MBF_2$ has a slightly more complex geometric visualization.

$$MBF_2 = \{(x_1, x_2, x_3, x_4) \in [0, 1]^4 | x_1 \leq \min(x_2, x_3), x_4 \geq \max(x_2, x_3)\}$$

$F_n$ can be understood as a function over $k$-space to $[0, 1]$, where $k = n \times 2^d$ and $d$ is the dimension of each MBF.
Let $S$ be a convex set. A function $f : S \rightarrow \mathbb{R}$ is quasi-concave if for each $a, b \in S$, $f(a + tb) \geq \min(f(a), f(b)) \ \forall t \in [0, 1]$.

The analog of $F_n$ in a 2-dimensional plane is a piece-wise linear function. In reality it is a piece-wise planar graph; $F_n$ is almost linear except at the points at which the min function changes its nature.
If a function is quasi-concave on its domain, then it is easy to see that the function can only take its minima at the extreme points of the convex set (points which do not lie along any line contained in the convex set).

Thus verifying non-negativity of $F_n$ across its domain would reduce to checking non-negativity at its extreme points.

An easy way to check quasi-concavity is to fix $n-1$ functions in $F_n$ and vary the $n^{th}$ function. In this case, the convex set $S$ is the set of all $d$-dimensional MBFs.
We began testing the quasi-concave property for $F_3$ but ended up finding a counter-example (Here $F_3$ is over one-dimensional MBFs).

**Figure:** $F_3$ as a function of $g$ ($f_0=[0.6,0.6], f_1=[0.1,1]$)
We began testing the quasi-concave property for $F_3$ but ended up finding a counter-example (Here $F_3$ is over one-dimensional MBFs).

Figure: $F_3$ as a function of $t$ ($f_0=[0.6,0.6]$, $f_1=[0.1,1]$, $g_0=[0,1]$, $g_1=[0.9,0.9]$)
$F_3$ fails the quasi-concavity test which means that it may have minima in its non-extreme points. These putative minima may be of two kinds:

1. **Zero minima**: Minima at which $F_3$ is zero
2. **Non-zero minima**: Minima at which $F_3$ is positive

There are numerous examples of zero minima; the simplest one is two of the functions being identically zero while the third one can be any function belonging to the convex set. Then $F_3$ is zero and the point is a non extreme point.

Siddhartha and I began searching for an algorithm that could generate these non-extreme non-zero minima.
Algorithm

This is the algorithm which given a random initial point, tries to decrease $F_3$ along connected points in the domain until it either terminates with a zero minima or a non-zero minima. This algorithm runs for $F_3$ for d-dimensional MBFs. Let $S \subset [0,1]^k$ denote the space of $n$ many d-dimensional MBFs ($k = n \times 2^d$).

1. Begin with a random point $p$ in $S$. $p$ is a $k$-tuple.
2. Repeat until $p$ does not change over $k$ successive iterations.
   1. For each co-ordinate of $p$:
      1. Compute $F_3(p_{1\_up})$ and $F_3(p_{1\_down})$
      2. $p = p_{1\_up}$ if $F_3(p_{1\_up}) \leq F_3(p_{1\_down})$ and $p = p_{1\_down}$ otherwise

The $p$ obtained from the previous steps is used for the next part of the algorithm on the next slide.
**Algorithm**

1. Repeat until $p$ does not change over $n$ successive iterations.
   1. For each of the $n$ functions:
      1. Partition the function’s co-ordinates, $\{1, \ldots, 2^d\}$, into subsets based on whether $p$ has the same value for indices of the subset.
      2. For each set in the partition:
         1. Compute $F_3(p^{2\text{up}})$ and $F_3(p^{2\text{down}})$
         2. $p = p^{2\text{up}}$ if $F_3(p^{2\text{up}}) \leq F_3(p^{2\text{down}})$ and $p = p^{2\text{down}}$ otherwise

After running the above algorithm a large number of times for $d = 1$ and 2 the algorithm always terminated at a zero minima. The mechanism used by the algorithm to reduce points to zero minima motivated me to search for a proof for showing that $F_3$ over one-dimensional MBFs has no non-zero minima.

Understanding the finer workings of the algorithm might provide a clue for showing that $F_3$ over all MBFs has no non-zero minima (no restriction on dimension of MBFs).
Proposition: $F_3$ over 1-dimensional MBFs has no non-zero minima.

Sketch of Proof:

- We prove the proposition by contradiction. Suppose $F_3$ does have a non-zero minima, say $(f_1, f_2, g_1, g_2, h_1, h_2)$.
- Since this point belongs to our convex set $S$ we have the following restrictions: $0 \leq f_i, g_i, h_i \leq 1$ and $f_1 \leq f_2, g_1 \leq g_2, h_1 \leq h_2$.
- Despite all these restrictions, there is still freedom in the linear ordering of the sets $\{f_1, g_1, h_1\}$ and $\{f_2, g_2, h_2\}$.
- By the symmetric nature of $F_3$ ($F_3(f, g, h) = F_3(g, f, h) = ..$), w.l.o.g. we can also assume $f_1 \leq g_1 \leq h_1$
- This leaves us with six cases based on the ordering of $\{f_2, g_2, h_2\}$
Sketch of Proof

- The co-ordinate wise orderings are important since they give an explicit formula for $F_3$. By looking at the sign of the coefficients of these co-ordinates we can claim that these co-ordinates must in fact be 0 or 1.

- Reductions of this type result in finally arriving at a contradiction: Step by step we impose that certain co-ordinates take specific values like 0 or 1 (the violation of which would contradict minimality of the point) and finally show that these steps lead to a zero minima, thus contradicting our initial assumption that the point was a non-zero minima.

- By considering these 6 broad cases, this argument holds true in general for any point belonging to $S$. 

Studying the nature of the minima of $F_n$ is an important step in attempting to show that $F_n$ is non-negative.

I have successfully showed that $F_3$ over one-dimensional MBFs has no non-zero minima.

Studying how the algorithm eventually reaches a zero minima for $F_3$ over any MBF will provide an insight into a potential argument for a formal proof.

A correlation inequality in 3 monotone functions has applications in probability theory, combinatorics, stochastic processes and statistical mechanics.

It would be exciting to see a stronger version of $E_3$ being applied in areas like uniform random spanning tree measures, symmetric exclusion processes, random cluster models (with $q < 1$), balanced and Rayleigh matroids.
References

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I would like to thank Prof. Siddhartha Sahi for his time, effort, encouragement and mentor-ship over the last three months.

I am grateful to Prof. Lazaros Gallos and DIMACS for giving me the opportunity to work on this project in Rutgers University.

This work is supported by the Rutgers Department of Mathematics. [NSF Grant DMS-2001537].