Ramsey Numbers of 0-1 Matrices

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Ramsey’s theorem

Ramsey, 1928

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0-1 matrices

\[
\begin{pmatrix}
1 \\
1 \\
1 \\
1
\end{pmatrix}
\]
Identity matrix

\[
\begin{pmatrix}
1 \\
1 \\
1 \\
1
\end{pmatrix}
\]
Permutation matrix

\[
\begin{pmatrix}
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1
\end{pmatrix}
\]
Matrix of ones.

\[
\begin{pmatrix}
1 \\
1 \\
1 \\
1
\end{pmatrix}
\]
L matrix.
What is a submatrix?

\[
\begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix}
\]

is a submatrix of

\[
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 & 1
\end{pmatrix}
\]
What is a submatrix?

\[
\begin{pmatrix}
1 \\
1 & 1
\end{pmatrix}
\]

is a submatrix of

\[
\begin{pmatrix}
& & & \\
\downarrow & \downarrow & \downarrow & \\
\rightarrow 1 & 1 \\
\rightarrow 1 & 1 \\
\rightarrow 1 & 1
\end{pmatrix}
\]
Problem

Question

Given a $k \times k$ matrix $A$, what is the smallest number $N = R(A)$ such that for any coloring of cells of a $N \times N$ table $C$ with two colors there always is a subtable $B \subseteq C$ of size $k \times k$, such that all positions $(x, y)$ in $B$ with $A_{x,y} = 1$ have the same colour.
Example

For $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$
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What is known

For the identity matrix $I_k$ we have $R(I_k) = 2^k - 1$ (pigeonhole on diagonal).

For the all-ones matrix $O_k$ we have $R(O_k) \approx 2^k$.

For any permutation matrix $P_k$ it is true that $R(P_k) \leq k^2$.

There are permutation matrices $P'_k$ such that $R(P'_k) \geq k^2 \log_2 k$. 
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- There are permutation matrices $P'_k$ such that $R(P'_k) \geq \frac{k^2}{\log^2 k}$. 
Our goals

Get better estimates for some more restricted classes of permutation matrices.
Get estimates for some other classes.
Investigate the boundary between $\mathbb{R}(M)$ being polynomial and super-polynomial.
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