Ramsey Numbers of 0-1 Matrices

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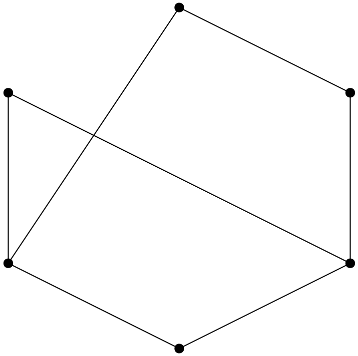

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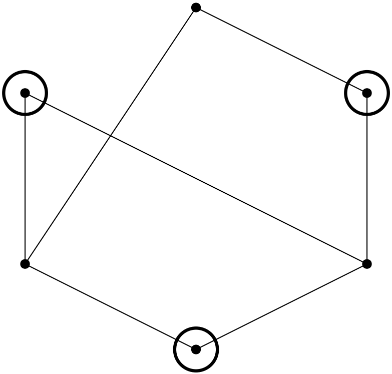

Frank P. Ramsey

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0-1 matrices

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

Identity matrix

$$\begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}$$

Permutation matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Matrix of ones.

$$\begin{pmatrix} 1 & & & \\ 1 & & & \\ 1 & & & \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

L matrix.

What is a submatrix?

$$\begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix}$$

is a submatrix of

$$\begin{pmatrix} 1 & & 1 & \\ & 1 & & \\ 1 & & & 1 \\ & & 1 & 1 \end{pmatrix}$$

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$$\begin{array}{ccccc}
 & \downarrow & & & \downarrow \\
\rightarrow & 1 & & 1 & \\
 & & 1 & & \\
\rightarrow & 1 & & & 1 \\
 & & & 1 & 1
\end{array}$$

Problem

Question

Given a $k \times k$ matrix A, what is the smallest number $N = R(A)$ such that for any coloring of cells of a $N \times N$ table C with two colors there always is a subtable $B \subseteq C$ of size $k \times k$, such that all positions (x, y) in B with $A_{x,y} = 1$ have the same colour.

Example

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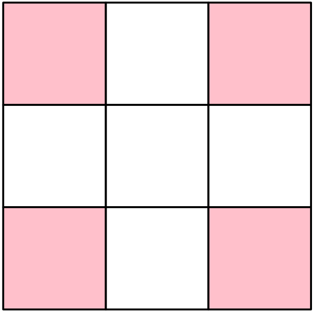

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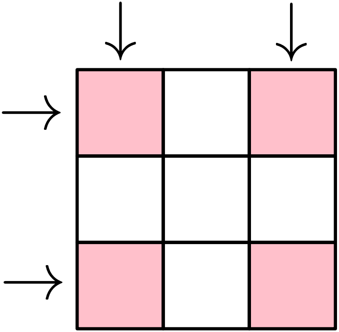

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- There are permutation matrices P'_k such that $R(P'_k) \geq \frac{k^2}{\log^2 k}$.

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- Investigate the boundary between $R(M)$ being polynomial and super-polynomial.

Acknowledgements

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