Max-Cut in the 2-Party Communication Model

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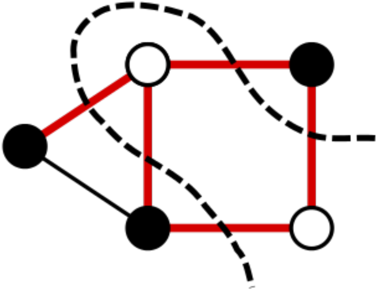
The Weighted Max-Cut Problem

- Cut S on a graph $G = (V, E)$: a partition of the set of vertices V into disjoint subsets S, T
- Size of a cut for an undirected graph G with weight function $w : E \to \mathbb{R}$ is $\delta(S) = \sum_{(u,v)\in E:\ u\in S, v\in T} w(u, v)$
- Problem: Can we find a cut S of maximum size (optimization) or of size $k > 1$ (decision)?



The 2-Party Communication Model

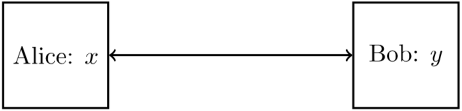


Figure 1: Alice and Bob computing f(x,y)

- Two-way communication: Alice and Bob both send messages to compute $f(x, y)$
- Communication cost: how many bits of communication a protocol requires
- Communication Complexity (CC): the min communication cost over all protocols that compute $f(x, y)$ correctly (deterministic) or correctly with probability $2/3$ (randomized)

Weighted Max-Cut Communication Problem

Problem:

- Undirected graph $G = (V, E)$, with weight function $w : E \to \mathbb{Z}^+$
- Alice receives E_A and Bob receives E_B such that $E = E_A \cup E_B$ - also $w(e)\ \forall e \in E_A$ and $w(e)\ \forall e \in E_B$ respectively
- Alice and Bob communicate through some protocol π
- Goal: find a maximum cut on G

Question: What is the communication complexity of finding a maximum cut - i.e. how many bits of communication are required?

Previous Results

- Trivial upper bound of $O(n^2 \log n)$ for Weighted Max Cut and $O(n^2)$ for weights 1
- Trivial lower bound of $\Omega(n \log n)$ for Weighted Max Cut and $\Omega(n)$ for weights equal to 1
- Lower bound of $\Omega(n^2)$ with one-way communication for weighted Max Cut [Zel11]

Weighted Max-Cut Lower Bound Result

Theorem: For an undirected graph $G = (V, E)$ with n vertices and $\Omega(n^2)$ edges, the Weighted Max-Cut communication problem requires $\Omega(n^2)$ communication complexity.

Established through a chain of reductions:
Disjointness $\leq$ 3-Coloring $\leq$ NAE 3-SAT $\leq$ Max-Cut

Necessary Communication Problems: Disjointness

- Alice and Bob have subsets A and $B \subseteq [k] = \{1, \ldots, k\}$
- 2 cases possible: $A \cap B = \{i\}$ for some $i \in [k]$ or $A \cap B = \emptyset$
- Output 1 if $A \cap B = \{i\}$ and 0 if $A \cap B = \emptyset$
- Previous result: $\Omega(k)$ CC lower bound [BJKS04, KS92, Raz92]

Necessary Communication Problems: 3-Coloring

- Alice and Bob have E_A and E_B of an undirected graph $G = (V, E = E_A \cup E_B)$
- Output a valid coloring of every vertex with one of 3 permitted colors (no monochromatic edges)
- Output 0 if no such coloring exists
- Let $n = |V|$
- Lemma: There is a CC lower bound of $\Omega(n^2)$ for 3-Coloring.

Reduction 1: Disjointness $\leq$ 3-Coloring

Construct an instance of 3-Coloring using 3 gadgets:

- Color Gadget
- Color-Coordinator Gadget
- Odd-Cycle Gadget

Reduction 1: Color Gadget

- 3 vertices forming a triangle - each must be colored differently
- Any vertex colored the same as vertex RED is said to be colored with the color red (etc.)

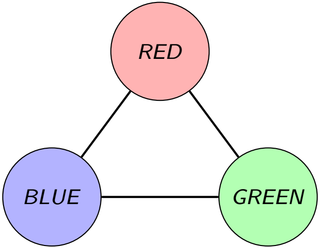


Figure 2: Visual representation of the Color Gadget

Reduction 1: Color-Coordinator Gadget

Color-Coordinator(u, v) for $u, v \in V$ creates a subgraph of $G = (V, E)$ such that:

- $O(1)$ new vertices are added
- u, v must be colored with the same color

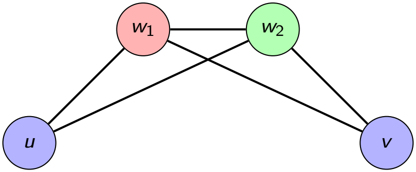


Figure 3: A sample construction of the Color-Coordinator Gadget

Reduction 1 : Odd-Cycle Gadget

Odd-Cycle($c1,\ c2,\ t,\ u_1,\ \ldots,\ u_t$) where $t \in \mathbb{Z}^+$, $u_1, \ldots, u_t \in V$, and $c1, c2 \in \{RED, GREEN, BLUE\}$ creates a subgraph of $G = (V, E)$ such that:

- $O(t)$ new vertices are added
- $\exists i \in [t]$ such that u_i is colored the same as $c2$
- $\forall i \in [t]$ exists a 3-coloring of the subgraph such that $\{u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_t\}$ are colored the same as $c3$

Odd-Cycle Gadget (continued)

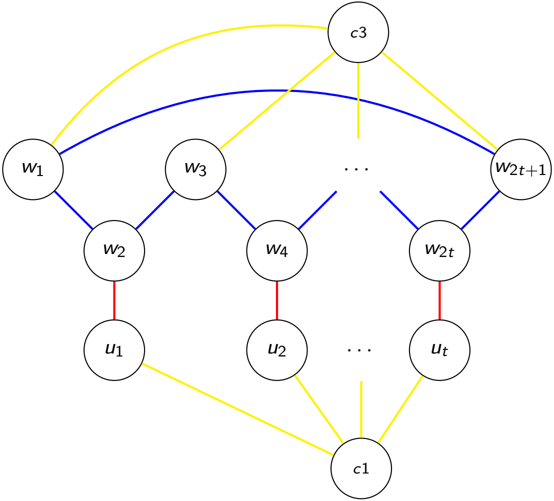


Figure 4: A sample construction of the Odd-Cycle Gadget

Reduction 1: 3-Coloring Construction

DISJ: Alice has $A = \{(i,j) : i,j \in [t]\}$, Bob has $B = \{(i,j) : i,j \in [t]\}$

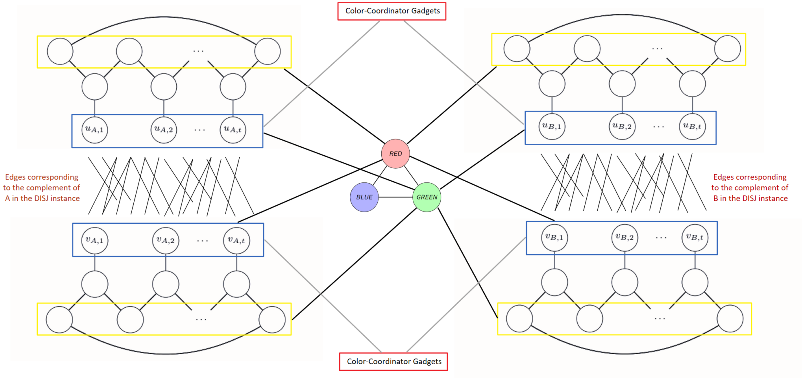


Figure 5: 3-Coloring instance corresponding to the given instance of DISJ

Conclusion

- Showed Disjointness $\leq$ 3-Coloring such that 3-Coloring has $O(t)$ vertices for t^2 elements in Disjointness
- $\Omega(t^2)$ Disjointness CC lower bound gives $\Omega(n^2)$ 3-Coloring CC lower bound
- We apply and alter some reductions in a white-box way, getting the chain 3-Coloring $\leq$ NAE 3-SAT $\leq$ Max Cut
  - $\Omega(n^2)$ CC lower bound for Weighted Max Cut
- Current goal: continue working on lower bound for Max Cut with weights of 1

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