On the Toric Varieties Associated with Bicolored Metric Trees

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Background in Algebraic Geometry

• An **affine variety** is a zero set of some polynomials in \( C[x_1, \ldots, x_n] \).

• In general, we can replace \( C \) by an **algebraically closed field** \( K \) and define an affine variety to be a zero set of some polynomials in \( K[x_1, \ldots, x_n] \).

• Given an **ideal** \( I \) of \( C[x_1, \ldots, x_n] \), denote by \( V(I) \) the zero set of the polynomials in \( I \). Then \( V(I) \) is an affine variety.
Background in Algebraic Geometry

- **Hilbert’s Lemma**: For any proper ideal $I$ in $C[x_1, \ldots, x_n]$, the corresponding variety $V(I)$ is nonempty.

- Given a variety $V$ in $C^n$, denote $I(V)$ to be the set of polynomials in $C[x_1, \ldots, x_n]$ which vanish in $V$.

- Define the quotient ring $C[V] = C[x_1, \ldots, x_n] / I(V)$ to be the coordinate ring of the variety $V$.

- From the Hilbert’s Lemma, it follows that there is a natural bijective map from the set of maximal ideals of $C[V]$ to the points in the variety $V$. 
Background in Algebraic Geometry

• We can write

\[ \text{Spec}(C[V]) = V \]

where Spec(C[V]) is the spectrum of the ring C[V], i.e. the set of maximal ideals of C[V].

• **Theorem:** \( V = \text{Spec}(C[V]) = \text{Hom}(C[V], C) \)

• Example: If \( V = C^n \), then \( I(V) = \{0\} \) and \( C[V] = C[x_1, \ldots, x_n] \).
  
  Since the ideals \((x_1 - a_1, \ldots, x_n - a_n)\) are all the maximal ideals of \( C[x_1, \ldots, x_n] \), we obtain
  
  \[ \text{Spec}(C[x_1, \ldots, x_n]) = C^n. \]
Background in Algebraic Geometry

• In $\mathbb{C}^n$, we define the **Zariski topology** as follows: a set is closed if and only if it is an affine variety.

• Example: In $\mathbb{C}$, the closed set in the Zariski topology is a finite set.

• Open sets in Zariski topology tend to be “large”.
• An irreducible variety $V$ is a variety that cannot be written as union of proper subvarieties $V = V_1 \cup V_2$.

• A variety $V$ is irreducible if and only if its coordinate ring $\mathbb{C}[V] = \mathbb{C}[x_1, \ldots, x_n] / I(V)$ is prime.
Constructing Toric Varieties from Cones

• A convex cone $\sigma$ in $\mathbb{R}^n$ is defined to be

$$\sigma = \{ r_1 v_1 + \ldots + r_k v_k \mid r_i \geq 0 \}$$

where $v_1, \ldots, v_n$ are given vectors in $\mathbb{R}^n$

• Denote the standard dot product $<,>$ in $\mathbb{R}^n$.

• Define the dual cone of $\sigma$ to be the set of linear maps from $\mathbb{R}^n$ to $\mathbb{R}$ such that it is nonnegative in the cone $\sigma$.

$$\sigma^\vee = \{ u \in \mathbb{R}^n \mid <u, v> \geq 0 \text{ for all } v \in \sigma \}$$
Constructing Toric Varieties from Cones

- A cone $\sigma$ is **rational** if its generators $\nu_i$ are in $\mathbb{Z}^n$.

- A cone $\sigma$ is **strongly convex** if it doesn’t contain a linear subspace.

- From now on, all the cones $\sigma$ we consider are rational and strongly convex.

- Denote $M = \text{Hom}(\mathbb{Z}^n, \mathbb{Z})$, i.e. $M$ contains all the linear maps from $\mathbb{Z}^n$ to $\mathbb{Z}$.
Constructing of Toric Varieties from Cones

- **Gordon’s Lemma**: $\sigma^v \cap M$ is a finitely generated semigroup.

- Denote $S_\sigma = \sigma^v \cap M$. Then $C[S_\sigma]$ is a finitely generated commutative $C$–algebra.

- Define

  $$ U_\sigma = \text{Spec}(C[S_\sigma]) $$

  Then $U_\sigma$ is an affine variety.
Constructing Toric Varieties from Cones

• If $\tau \subset \sigma \subset R^n$ is a face, we have $C[S_\sigma] \subset C[S_\tau]$ and hence obtain a natural gluing map $U_\tau \to U_\sigma$. Thus all the $U_\tau$ fit together in $U_\sigma$.

Remarks

• Consider the face $\tau = \{0\} \subset \sigma$. Then $U_\tau = (C^*)^n = T_n$ here $C^* = C \setminus \{0\}$. Thus for every cone $\sigma \subset R^n$, there is an embedding $T_n \to U_\sigma$.

• There is a natural bijective correspondence between points in $U_\sigma$ and elements in $Hom(S_\sigma, C)$.

$$U_\sigma = Hom(S_\sigma, C).$$
Example: Consider $\sigma$ to be generated by $e_2$ and $2e_1 - e_2$. Then the dual cone $\sigma^v$ is generated by $e_1^*$ and $e_1^* + 2e_2^*$. However, $S_\sigma = \sigma^v \cap M$ is generated by $e_1^*$, $e_1^* + e_2^*$ and $e_1^* + 2e_2^*$. 
Constructing Toric Varieties from Cones

Let \( x = e_1^* \), \( xy = e_1^* + e_2^* \), \( xy^2 = e_1^* + 2e_2^* \).

Hence \( S_\sigma = \{ x^a(xy)^b(xy^2)^c = x^{a+b+c}y^{b+2c} \mid a, b, c \in \mathbb{Z} \geq 0 \} \)

Then \( C[S_\sigma] = C[x, xy, xy^2] = C[u, v, w] / (v^2 - uw) \).

Thus \( U_\sigma = \{ (u,v,w) \in C^3 \mid v^2 - uw = 0 \} \)
Constructing Toric Varieties from Cones

- If we denote $\tau_0$, $\tau_1$, $\tau_2$ the faces of the cone $\sigma$. Then

  $$U_{\tau_0} = (C^*)^2 \rightarrow \{ (u,v,w) \in (C^*)^3 \mid v^2 - uw = 0 \}$$

  $$U_{\tau_1} = (C^*) \times C \rightarrow \{ (u,v,w) \in (C)^3 \mid v^2 - uw = 0, u \neq 0 \}$$

  $$U_{\tau_2} = (C^*) \times C \rightarrow \{ (u,v,w) \in (C)^3 \mid v^2 - uw = 0, w \neq 0 \}$$
Action of the Torus

- Recall that $V = \text{Spec}(C[V]) = \text{Hom}(C[V], C)$.

- From this, we can define the torus $T_n$ action on $U_\sigma$ as follows:
  1. $t \in T_n$ corresponds to $t^* \in \text{Hom}(S_{\{0\}}, C)$
  2. $x \in U_\sigma$ corresponds to $x^* \in \text{Hom}(S_\sigma, C)$
  3. Thus $t^* \cdot x^* \in \text{Hom}(S_\sigma, C)$ and we denote $t \cdot x$ the corresponding point in $S_\sigma$.
  4. We have the following group action:
     $$T_n \times U_\sigma \rightarrow U_\sigma$$
     $$ (t, x) \mapsto t \cdot x$$

- Torus varieties: contains $T_n = (C^*)^n$ as a dense subset in Zariski topology and has a $T_n$ action.
Orbits of the Torus Action

• The toric variety $U_\sigma$ is a disjoint union of the orbits of the torus action $T_n$.

• Given a face $\tau \subset \sigma$. Denote by $N_\tau \subset \mathbb{Z}^n$ the lattice generated by $\tau$.

• Define the quotient lattice $N(\tau) = \mathbb{Z}^n / N_\tau$ and let $O_\tau = T_{N(\tau)}$, the torus of the lattice $N(\tau)$. Hence $O_\tau = T_{N(\tau)} \cong (C^*)^k$.

• There is a natural embedding of $O_\tau$ into an orbit of $U_\sigma$.

• **Theorem** There is a bijective correspondence between orbits and faces. The toric variety $U_\sigma$ is a disjoint union of the orbits $O_\tau$ where $\tau$ are faces of the cones.
Torus Varieties Associated With Bicolored Metric Trees

- Bicolored Metric Trees

\[ V(T) = \{ (x_1, x_2, x_3, x_4, x_5, x_6 \in C^6 \mid x_1x_3 = x_2x_5, x_3 = x_4, x_5 = x_6 \} \]
Toric Varieties Associated With Bicolored Metric Trees

• **Theorem** $V(T)$ is an affine toric variety.
Description of the Cones for the Toric Varieties

• Given a bicolored metric trees $T$.
• We can **decompose** the tree $T$ into the sum of smaller bicolored metric trees $T_1, \ldots, T_m$. 

![Diagram of a tree decomposition](image-url)
Description of the Cones for the Toric Varieties

• We give a description of the cone $\sigma(T)$ by induction on the number of nodes $n$.

• For $n = 1$, we have

Thus $\sigma(T)$ is generated by $e_1$ in $C$.

• Suppose we constructed the generators for any trees $T$ with the number of nodes less than $n$.

• Decompose the trees into smaller trees $T_1, \ldots, T_m$. For each tree $T_k$, denote by $G_k$ the constructed set of generators of $\sigma(T_k)$.
Description of the Cones for the Toric Varieties

• Then the set of generators of the cone $\sigma(T)$ is

$$G(T) = \{ e_n + \alpha_1 (z_1 - e_n) + \ldots + \alpha_m(z_m - e_n) | z_k \in G_i, \alpha_k \in \{0,1\} \}$$

**Remarks**

• The cone $\sigma(T)$ is $n$ dimensional, where $n$ is the number of nodes.

• The elements in $G(T)$ corresponds bijectively to the rays (i.e. the 1-dimensional faces) of the cone $\sigma(T)$

• If the dimension of $\sigma(T_k)$ is $n_k$, then the dimension of $\sigma(T)$ is $(n_1 + 1) \ldots (n_m + 1)$. 
Description of the Cones for the Toric Varieties

Example: \( T = T_1 + T_2 + T_3 \)

\[
G(T_1) = \{e_1\}, \ G(T_2) = \{e_2\}, \ G(T_3) = \{e_3\}
\]

\[
G(T) = \{e_1, e_2, e_3, e_1 + e_2 - e_4, e_1 + e_3 - e_4, e_2 + e_3 - e_4, e_1 + e_2 + e_3 - 2e_4\}
\]
Weil Divisors

- A prime Weil divisor $D$ of a variety $V$ is an irreducible subvariety of codimensional 1.

- A Weil divisor is an integral linear combination of prime Weil divisors.

- Given a bicolored metric tree $T$ with variety $V(T)$. We care about toric-invariant prime Weil divisors, i.e. $T_n(D) \subset D$.

- **Theorem:** $D = \text{clos}(O_\tau)$ for some 1 dimensional face $\tau$.

- There is a natural bijective correspondence between the set of prime Weil divisor and the set of generators $G(T)$.
Weil Divisors

We can also list all the toric-invariant prime Weil divisors as follows:

• Let $x_1, \ldots, x_N$ be all the variables we label to the edges of $T$.

• We call a subset $Y = \{y_1, \ldots, y_m\}$ of $\{x_1, \ldots, x_N\}$ complete if it has the property that $x_k = 0$ if and only if $x_k \in Y$ when we set $y_1 = \ldots = y_m = 0$.

• A complete subset $Y$ is called minimally complete if it doesn’t contain any other complete subset.
Weil Divisors

• **Lemma** $Y$ is a minimally complete subset if and only if the unique path from each colored point to the root contains exactly one edge with labelled variable $y_k$.

• **Theorem** $D$ is a toric-invariant prime Weil divisor if and only if $D = \{(x_1, \ldots, x_N) \in V(T) | x_k = 0, \forall x_k \in Y\}$ for some minimally complete $Y$.

• **Corollary** There is a natural bijective correspondence between the set of toric-invariant prime Weil divisors and the set of minimally complete subsets.
Weil Divisors

We can describe the correspondence map in theorem by induction on the number of nodes $n$.

- For $n = 1$, it is simple.

- Decompose the tree $T$ into $T_1, \ldots, T_m$ and let $x_1, \ldots, x_m$ be the labelled variables on the principal branches (which are now the roots of $T_1, \ldots, T_m$).

- Let $D$ be a toric-invariant prime Weil divisor and $Y$ be its corresponding minimally complete set. Let $\chi_Y$ be the character function on $Y$. 
Weil Divisors

- If $x_k \notin Y$ for some $1 \leq k \leq m$, then $D$ induces naturally a prime Weil divisor $D_k$ of $V(T_k)$, which by induction corresponds to an element $z_k$ in $G(T_k)$.

- If $x_k \in Y$ for some $1 \leq k \leq m$, define $z_k = 0$.

- $D$ corresponds to $e_n + (1 - \chi_Y(x_1))(z_1 - e_n) + \ldots + (1 - \chi_Y(x_m))(z_m - e_n)$. 
Weil Divisors

Example: There are 4 prime Weil divisors $D_{12}$, $D_{156}$, $D_{234}$, $D_{3456}$ which corresponds to the minimal complete sets $Y_{12}$, $Y_{156}$, $Y_{234}$, $Y_{3456}$.

Decompose the tree into two trees $T_1$ and $T_2$. Then $G(T_1) = \{e_1\}$ and $G(T_2) = \{e_2\}$. Thus $D_{12}$, $D_{156}$, $D_{234}$, $D_{3456}$ corresponds to $e_3$, $e_2$, $e_1$, $e_2 + e_3 - e_1$. 
Cartier Divisors

• Given a variety $V$. For every point $p \in V$, we define the local ring $O_p = \{f(x)/g(x) \mid g(p) \neq 0, f, g \text{ are polynomials}\} \subset C(V)$.

• It is a theorem that if $V$ is a normal affine variety, then $O_p$ is a discrete valuation ring.

• Hence given any subvariety $D$ of codimension 1 of $V$ and $p \in D$, we can define $ord_{D,p}(f)$ for every function $f \in C(V)$.

• Informally speaking, $ord_{D,p}(f)$, determines the order of vanishing of $f$ at $p$.

• It turns out that $ord_{D,p}$ doesn’t depend on $p \in V$. 
Theorem  The toric variety $V(T)$ is normal for any tree $T$.

It makes sense to talk about $ord_D$ for each prime Weil divisor $D$ (not necessarily toric-invariant).

For each $f$, define

$$div(f) = \sum_D (ord_D(f)).D$$

If $ord_D(f) = 0$ for every non toric-invariant prime Weil divisor, we call $div(f)$ a toric-invariant Cartier divisors.
Cartier Divisors

- **Theorem** \( \text{div}(f) \) is toric-invariant if and only if \( f \) is a fraction of two monomials.

- Recall that we have constructed a natural correspondence between the prime Weil divisors of \( V(T) \) and the elements in \( G(T) \).

- Call \( D_1, \ldots, D_N \) the prime Weil divisors of \( V(T) \) and \( v_1, \ldots, v_N \) the corresponding elements in \( G(T) \).

- Recall that if the tree \( T \) has \( n \) nodes, then \( v_1, \ldots, v_N \) are vectors in \( \mathbb{Z}^n \).
Cartier Divisors

• **Theorem**  A Weil divisor $D = \sum a_i D_i$ is Cartier if and only if there exists a map $\varphi \in \text{Hom}(\mathbb{Z}^n, \mathbb{Z})$ such that $\varphi(v_i) = a_i$.

Example  The prime Weil divisors of $T$ are $D_{12}$, $D_{156}$, $D_{234}$, $D_{3456}$ corresponds to $e_3$, $e_2$, $e_1$, $e_2 + e_3 - e_1$. Hence $aD_{12} + bD_{156} + cD_{234} + dD_{3456}$ is Cartier if and only if $a + d = b + c$. 
Cartier Divisors

• We can describe the generators of the set of Cartier divisors as follows:

1) Denote the nodes of $T$ to be the point $P_1, \ldots, P_n$.

2) Call $Y_1, \ldots, Y_N$ the corresponding minimal complete set of the prime Weil divisors $D_1, \ldots, D_N$.

3) For each node $P_k$, let $\phi_k$ be a map defined on \{\(D_1, \ldots, D_N\}\) such that $\phi_k(D_i) = 0$ if there is no element in $Y_i$ in the subtree below $P_k$.

   Otherwise $1 - \phi_k(D_i) = \text{the number of branches of } P_k \text{ that doesn’t contain an element in } Y_i$. 
Cartier Divisors

• **Theorem** The set of Cartier divisors is generated by
  \[
  \{ \phi_k(D_j) D_j + \ldots + \phi_k(D_N) D_N \mid 1 \leq k \leq m \}
  \]

**Example:** The generators of the Cartier divisors of T are
\[
D_{12} - D_{3456}, \quad D_{34} + D_{3456}, \quad D_{156} + D_{3456}.
\]
Summary of Results

Description of toric-invariant Weil divisors and Cartier divisors of the torus variety associated with a bicolored metric tree.

Possible future plan
Reference

