Floer Cohomology on Surfaces

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Introduction

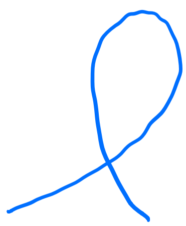
This is an introduction to constructing Floer cohomology on surfaces with genus greater than one. The following can be found in [1], [2] and is followed closely. The author makes no claims of originality.

Let Σ be a closed connected surface w/ genus greater than one.

An oriented immersed curve γ on Σ is unobstructed if

i) γ is not null-homotopic

ii) γ doesn't bound an immersed fishtail



fishtail

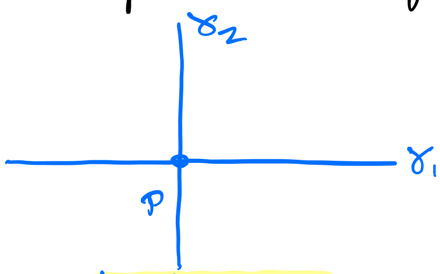
A pair (γ_1, γ_2) of unobstructed curves is admissible if

any Euler zero 2-chain between (γ_1, γ_2) has regions w/ positive and negative components

Basically, the curves cannot bound a cylinder

We will also assume our curves have a marked point

Locally we can always see a point $p \in \gamma_1 \cap \gamma_2$ as



local model

The degree of p, ϵ_p, is:

positive: if the standard orientation in $\mathbb{R}^2$ agrees w/ orientations on γ_1, γ_2 in local model

negative: if the standard orientation in $\mathbb{R}^2$ disagrees w/ orientations on γ_1, γ_2 in local model

The Floer chain complex of admissible pair (γ_1, γ_2) is the $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{Z}$-module

$$CF(\gamma_1, \gamma_2) = \sum_{\substack{p \in \gamma_1 \cap \gamma_2 \\ \epsilon_p = (-1)^{i+1}}}$$

w/ grading

even: ϵ_p negative

odd: ϵ_p positive

A **bigon** from p to q, $p, q \in \gamma_1 \cap \gamma_2$, is oriented immersion

$$\mathbb{B} \xrightarrow{u} \Sigma$$

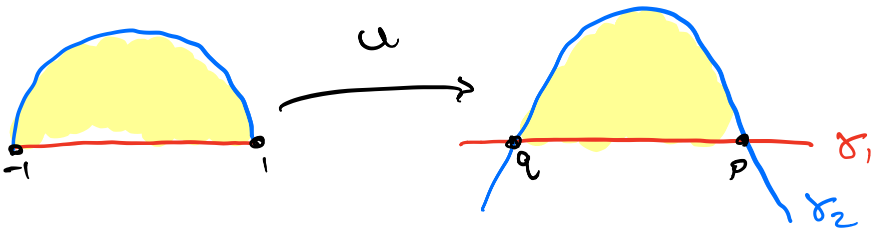
(upper half disk)

s.t.

i) $-1 \mapsto q$, $1 \mapsto p$

ii) p, q are convex corner points

iii) $u(\mathbb{B} \cap \mathbb{R}) \subset \gamma_1$, $u(\mathbb{B} \cap S^1) \subset \gamma_2$



Let $\mathcal{M}(q,p)$ be set of bigons from p to q.

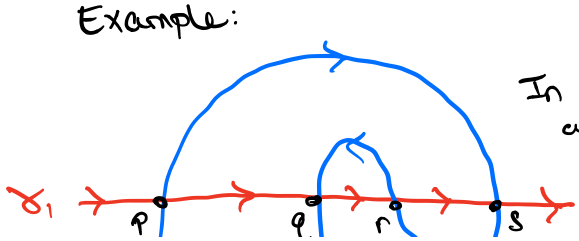
The **differential** is defined as

$$d(p) = \sum_{q,\, u \in \mathcal{M}(q,p)} (-1)^{s(u)} q$$

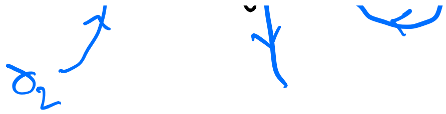
where $s(u)$ gets: $+1$ contribution if marked points on ∂u
$+1$ contribution if the orientation of $u(\mathbb{B} \cap S^1) \subset \gamma_2$ disagrees w/ given orientation on γ_2

We have that $d^2 = 0$. See [1] or [2] for the proof.

Example:



In some Σ w/ orientation and marked points not in picture



$CF^0(\gamma_1,\gamma_2) = \mathbb{Z}q + \mathbb{Z}s$

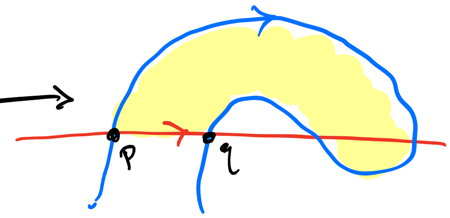
$CF^1(\gamma_1,\gamma_2)$ $\mathbb{Z}p + \mathbb{Z}r$

$d(q) = (-1)^{s(u)}p + (-1)^{s(v)}r$

no bigons from r to q

bigon from p to q

$d(q) = -p$

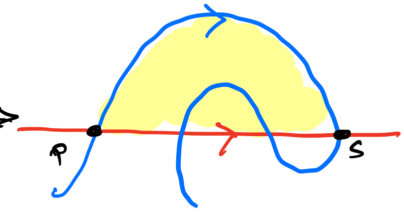


$d(s) = (-1)^{s(u)}p + (-1)^{s(v)}r$

no bigons from s to r

bigon from s to p

$d(s) = -p$



$d(p) = (-1)^{s(u)}q + (-1)^{s(v)}s$

no bigons from p to q or s

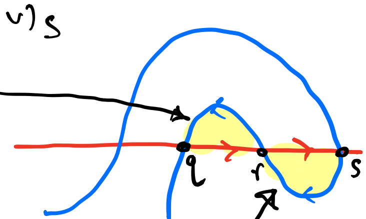
$d(p) = 0$

$d(r) = (-1)^{s(u)}q + (-1)^{s(v)}s$

bigon from r to q

bigon from r to s

$d(r) = q - s$



And thus satisfies $d^2 = 0$!

References

[1] M. Abouzaid. *On the Fukaya Categories of Higher Genus Surfaces*. 2006. arXiv: 0606598 [math.SG].

[2] H. Azam and C. Blanchet. *Fukaya Category of Surfaces and Mapping Class Group Action*. 2020. arXiv: 1903.11928 [math.GT].

Acknowledgements

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