The Novikov Covering

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Introduction

This is a brief overview of the material needed to define the Novikov covering of the space of paths between Lagrangians. The following can be found in [1], [2] and is followed closely. The author makes no claims of originality.

Outline

1. The Space of Paths Between Lagrangians
2. The Universal Cover of Ω
3. The Γ-Equivalence
4. The Novikov Covering of Ω

The Space of Paths

Let (L_0, L_1) be a pair of compact Lagrangian subamanifolds of (M, ω). Consider

$$\Omega(L_0, L_1) = \{[0,1] \xrightarrow{\ell} M : \ell(0) \in L_0, \ell(1) \in L_1\}. \tag{1}$$

By specifying a base path $\ell_0 \in \Omega(L_0, L_1)$ we get the connected component

$$\Omega(L_0, L_1; \ell_0). \tag{2}$$

Hence we may assume (L_0, L_1) are connected.

Why can we assume connected?

Choosing an ℓ_0 actually chooses connected components of (L_0, L_1) since

$$\ell_0(0) \in L_0$$
$$\ell_0(1) \in L_1.$$

So $\Omega(L_0, L_1; \ell_0)$ is the space of paths between these connected components.

We have the action 1-form given by

$$\alpha_\ell(Y) = \int_0^1 \omega(\dot{\ell}(t), Y(t))dt \tag{3}$$

for $Y \in T_\ell\Omega(L_0, L_1; \ell_0)$.

By viewing tangent vectors in $\Omega(L_0, L_1; \ell_0)$ as equivalence classes of curves we may view $Y \in T_\ell\Omega(L_0, L_1; \ell_0)$ as a vector field $Y(t)$ along ℓ.

Universal Cover

Consider set of pairs (ℓ, w) such that $\ell \in \Omega(L_0, L_1; \ell_0)$ and

$$[0,1]^2 \xrightarrow{w} M \tag{4}$$

subject to

$$\begin{cases} w(0,\cdot) = \ell_0 & w(1,\cdot) = \ell_1 \\ w(s,0) \in L_0 & w(s,1) \in L_1 \ \forall s \in [0,1] \end{cases}$$

If we consider w as a map $s \mapsto w(s,\cdot)$ then the fiber at ℓ of the universal cover $\Omega(L_0, L_1; \ell_0)$ can be represented by the set of path homotopy classes of w relative to its ends $s = 0, 1$.

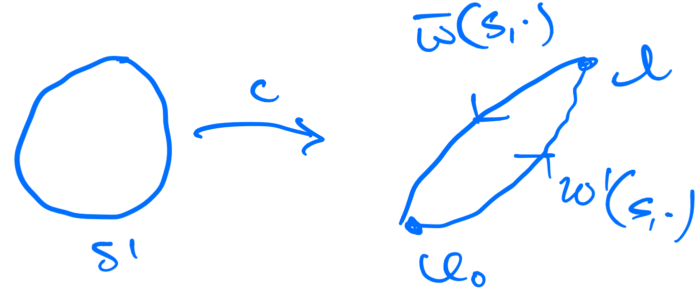
The Map w

Let $(\ell, w), (\ell, w')$ be two such pairs. Then the concatenation

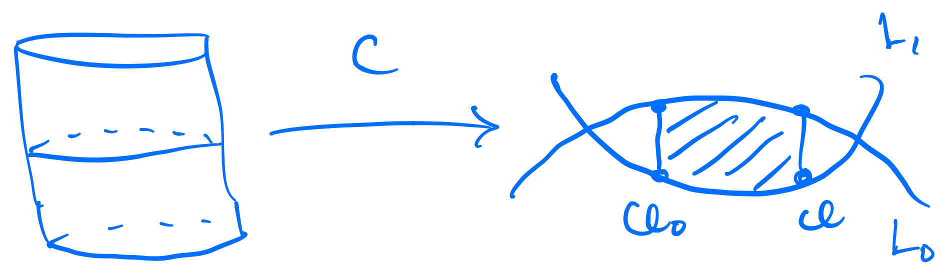
$$[0,1]^2 \xrightarrow{\overline{w}\# w'} M \tag{5}$$

induces maps:

$$S^1 \xrightarrow{c} \Omega(L_0, L_1; \ell_0).$$



$$S^1 \times [0,1] \xrightarrow{C} \mathcal{M}.$$



Some Homomorphisms

Since the symplectic area

$$I_\omega(c) = \int_C \omega \tag{6}$$

is independent of the homotopy of C we have a homomorphism

$$\pi_1(\Omega(L_0, L_1; \ell_0)) \xrightarrow{I_\omega} \mathbb{R}$$

Also we have that C associates a symplectic bundle pair

$$\mathcal{V}_C = C^* TM, \lambda_C = \coprod c_i^* TL_i \tag{7}$$

where $S^1 \xrightarrow{c_i} L_i$ is

$$c_i(s) = C(s,i) \quad , i = 0,1$$

Since $(\mathcal{V}_{\mathcal{C}}, \lambda_C)$ are independent of the homotopy of C this induces a homomorphism

$$\pi_1(\Omega(L_0, L_1; l_0)) \xrightarrow{I_\mu} \mathbb{Z}$$

Why are I_ω, I_μ independent of homotopy?

I_ω:

General fact that if A, B k-manifolds w/ $A \xrightarrow{f,g} B$ homotopic smooth maps & $\alpha \in \Omega^k(B)$ closed $\Rightarrow$

$$\int_A f^* \alpha = \int_A g^* \alpha$$

use Stoke's thm w/ w/ the homotopy

I_μ:

General fact that pullback of fibre bundles is invariant under homotopy

use the universal property

The Γ-Equivalence

We have that $\overline{w}\#w'$ induces maps c, C as before.

Definition

Two pairs $(\ell, w), (\ell, w')$ are said to be Γ-**Equivalent** if

$$I_\omega(\overline{w}\#w') = 0 = I_\mu(\overline{w}\#w') \tag{8}$$

The Novikov Covering of Ω

Definition

The **Novikov covering** $\tilde{\Omega}(L_0, L_1; \ell_0)$ is the set of Γ-Equivalent classes $[\ell, w]$.

We note ℓ_0 has a natural lift

$$[\ell_0, \tilde{\ell}_0] \text{ w/ } \tilde{\ell}_0(s,t) = \ell_0(t).$$

and so $\tilde{\Omega}(L_0, L_1; \ell_0)$ has a natural base point.

Deck Transformations of $\tilde{\Omega}$

Let $\Pi(L_0, L_1; \ell_0)$ denote the deck transformation group of $\tilde{\Omega}(L_0, L_1; \ell_0)$. Then I_ω, I_μ push down to homomorphisms

$$\Pi(L_0, L_1; \ell_0) \xrightarrow{E} \mathbb{R}$$

$$\Pi(L_0, L_1; \ell_0) \xrightarrow{\mu} \mathbb{Z}$$

$$E(g) = I_\omega[C] \quad ; \quad \mu(g) = I_\mu[C].$$

It can be shown $\Pi(L_0, L_1; \ell_0)$ is abelian since the map

$$\Pi(L_0, L_1; \ell_0) \xrightarrow{E \times \mu} \mathbb{R} \times \mathbb{Z}$$

is an injective gp. morphism.

Deck Transformations of $\tilde{\Omega}$ Cont.

What is $[C]$? Well

$$\pi(\ell_0, L_1; \ell_0) \cong \frac{\pi_1(\Omega(\ell_0, L_1; \ell_0))}{p_*(\pi_1(\tilde{\Omega}(\ell_0, L_1; \ell_0)))}$$

where p_* is projection of Novikov covering. So

g is some loop in $\Omega(\ell_0, L_1; \ell_0)$ up to composition

w/ loops satisfying $I_\omega = I_\mu = 0$. Then

$[C]$ is this class of g's identification.

References

[1] D. McDuff and D. Salamon. *Introduction to Symplectic Topology*. New York: Oxford University Press, 1995.

[2] K. Fukaya [et al.] *Lagrangian Intersection Floer Theory*. AMS/IP Studies in Advanced Mathematics, 2009.

Acknowledgements

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