Morse Homology

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This is a brief overview of the material needed to define the Morse complex. The following can be found in [1] and is followed closely. The author makes no claims of originality.
Outline

1 (Un)Stable Manifolds
2 The Space of Trajectories
3 The Space of Broken Trajectories
4 The Morse Complex
5 References and Acknowledgements
Let $V$ be a compact $n$-manifold with Morse function $f$ and adapted pseudo-gradient $X$. Suppose $a \in \text{Crit}(f)$ and $\varphi^s$ is the flow of $X$. Then:

**Definition**

The **stable manifold** of $a$ is

$$W^s(a) = \{ p \in V : \lim_{s \to \infty} \varphi^s(p) = a \}. \quad (1)$$

The **unstable manifold** of $a$ is

$$W^u(a) = \{ p \in V : \lim_{s \to -\infty} \varphi^s(p) = a \}. \quad (2)$$

Note it can be shown that

$$\dim W^u(a) = \text{codim} W^s(a) = \text{incl}(a)$$
Example

\[ \omega^s(a) = S^2 - \frac{3}{2} a_b^2 \]
\[ \omega^u(a) = \frac{3}{2} a_b^2 \]
\[ \omega^u(b) = S^2 - \frac{3}{2} a_b^2 \]
\[ \omega^s(b) = \frac{3}{2} a_b^2 \]

$S^2$ w/ natural height function
The pair \((f, X)\) satisfies the Morse-Smale condition if
\[
W^u(a) \cap W^s(b) \quad \forall a, b \in \text{Crit}(f)
\]
This implies
\[
\dim \left( W^u(a) \cap W^s(b) \right) = \text{incl}(a) - \text{incl}(b)
\]
Let \(\mathcal{M}(a, b) = W^u(a) \cap W^s(b)\). Equivalently,
\[
\mathcal{M}(a, b) = \bigcap_{p \in \mathcal{V}} \lim_{s \to -\infty} W^s(p) = a, \quad \lim_{s \to \infty} W^s(p) = b
\] (3)
Lemma

$\mathbb{R}$ acts on $M(a, b)$ by

$$(a, \phi) \mapsto \mathfrak{R} \mathcal{L}(\phi)$$

The action is free if $a \neq b$.

Thus $\mathcal{L}(a, b) = M(a, b)/\mathbb{R}$ is a manifold of dimension

$$\dim \mathcal{L}(a, b) = \text{ind}(a) - \text{ind}(b) - 1$$
Example

\[ L \subseteq (f(a), f(b)) \Rightarrow \]

\[ L(b,a) = M(b,a) \cap f^{-1}(a) \setminus \text{equator} \]

\[ L(b,a) \text{ is the set of trajectories from } b \text{ to } a. \]

\[ S^2 \text{ with natural height function} \]
A Compactification of $\mathcal{L}(a, b)$

Let

$$\overline{\mathcal{L}}(a, b) = \bigcup_{c_i \in \text{Crit}(f)} \mathcal{L}(a, c_1) \times \cdots \times \mathcal{L}(c_{q-1}, b).$$

(4)

By inducing a topology on $\overline{\mathcal{L}}(a, b)$ from the topology on $V$ we may show that the space is a compactification of $\mathcal{L}(a, b)$. 
A Topology on $\overline{L}(a, b)$: The Idea

Morse charts

Let $U = \bigcup \overline{U}_w$, $U^+ = \bigcup U^+_w$

$\lambda = (\lambda_1, \ldots, \lambda_2)$ is broken trajectory

Then

$\mu \in \mathcal{W}(\lambda, U^-, U^+)$

If $\mu$ is broken trajectory that leaves $\overline{U}_w$'s and enters $U^+_w$'s

"close" to $\lambda$.

Take $\mathcal{W}$'s as basis of topology on $\overline{L}(a, b)$.

Can be shown using sequences that $\overline{L}(a, b)$ is compact.
Let $R$ be a commutative ring, $\{A_i\}_{i=1}^n$ a set of $R$-modules, and $A_\ast$ the sequence

$$\cdots \rightarrow A_i \xrightarrow{\partial_i} A_{i-1} \rightarrow \cdots.$$  \hfill (5)

**Definition**

If $\partial_{i-1} \partial_i = 0$ then $A_\ast$ is called a *chain complex*.

Note that this is equivalent to

$$\operatorname{im} \partial_i \subset \ker \partial_{i-1}.$$  \hfill (6)
Since $\ker \partial_{i-1} \subset \text{im} \partial_i$ we have that the quotient group is well-defined:

**Definition**

The $i$-th *homology* group of the chain complex is the quotient group

$$H_i(A_\bullet) = \ker \partial_i / \text{im} \partial_{i+1}. \quad (7)$$
Let $V$ be a compact $n$-manifold with Morse function $f$ and adapted pseudo-gradient $X$ such that $(f, X)$ is Morse-Smale.

Denote by $C_k(f)$ the free $\mathbb{Z}/2$-vector space generated by the critical points of $f$ of index $k$.

**Definition**

The *Morse complex* is the sequence

$$
\cdots \rightarrow C_k(f) \xrightarrow{\partial_X} C_{k-1}(f) \rightarrow \cdots
$$

where $\partial_X$ is defined on a critical point as

$$
\partial_X(a) = \sum_{c \in \text{Crit}_{k-1}(f)} n_X(a,c) c \quad \text{with} \quad n_X(a,c) = (Z(a,b) \mod 2)
$$
The Morse Complex is a Complex

Let $a, b \in \text{Crit}(f)$.

1. If $\text{ind}(a) = \text{ind}(b) + 1$ then

   $\mathcal{F}(a,b) = \mathcal{L}(a,b)$ is a compact 0-manifold $\Rightarrow$ finite set of points.

   Hence $\eta_x(a,b)$ is well-defined.

2. If $\text{ind}(a) = \text{ind}(b) + 2$ then (let $\text{ind}(a) = k+1$, $\text{ind}(b) = k-1$)

   $\mathcal{F}(a,b)$ is a compact 1-manifold. Can be shown

   $\partial \mathcal{F}(a,b) = \bigcup_{c \in \text{Crit}_x(f)} \mathcal{L}(a,c) \times \mathcal{L}(c,b)$.

   But compact 1-manifolds have an even number of boundary points $\Rightarrow$

   $|\partial \mathcal{F}(a,b)| \equiv 0 \mod 2$
The Morse Complex is a Complex Cont.

Hence,

\[ \partial_x \circ \partial_x (a) = \sum_{c \in \text{Crit}_K(f)} \eta_x(a, c) c \]

\[ = \sum_{c \in \text{Crit}_K(f)} \eta_x(a, c) \left( \sum_{b \in \text{Crit}_{K-1}(f)} \eta_x(c, b) b \right) \]

\[ = \sum_{b \in \text{Crit}_{K-1}(f)} \left( \sum_{c \in \text{Crit}_K(f)} \eta_x(a, c) \eta_x(c, b) \right) b \]

\[ = 0. \]

So \( \partial_x \circ \partial_x = 0 \) \( \Rightarrow \) Morse Complex is a complex.
1. Morse homology is independent of the Morse-Smale pair \((f, X)\)
2. Morse homology is isomorphic to singular homology
3. Morse homology is a topological invariant
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