Morse Homology

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Introduction

This is a brief overview of the material needed to define the Morse complex. The following can be found in [1] and is followed closely. The author makes no claims of originality.

Outline

(Un)Stable Manifolds

Note: Convention s.t. $f \circ \varphi^s : \mathcal{M} \to \mathbb{R}$ is decreasing

Let V be a compact n-manifold with Morse function f and adapted pseudo-gradient X. Suppose $a \in \mathrm{Crit}(f)$ and φ^s is the flow of X. Then:

Definition

The *stable manifold of a* is

$$W^s(a) = \{p \in V : \lim_{s \to \infty} \varphi^s(p) = a\}. \tag{1}$$

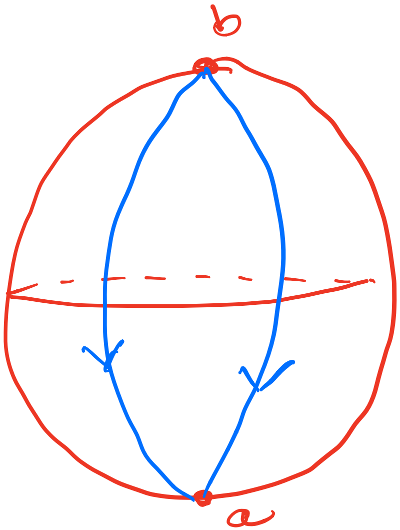
The *unstable manifold of a* is

$$W^u(a) = \{p \in V : \lim_{s \to -\infty} \varphi^s(p) = a\}. \tag{2}$$

Note it can be shown that

$$\dim W^u(a) = \operatorname{codim} W^s(a) = \operatorname{ind}(a)$$

Example



S^2 w/ natural height function

$W^s(a) = S^2 - \{b\}$

$W^u(a) = \{a\}$

$W^u(b) = S^2 - \{a\}$

$W^s(b) = \{b\}$

Morse-Smale Condition

The pair (f, X) satisfies the Morse-Smale condition if

$$W^u(a) \pitchfork W^s(b) \quad \forall a, b \in \mathrm{Crit}(f)$$

This implies

$$\dim\left(W^u(a) \pitchfork W^s(b)\right) = \mathrm{ind}(a) - \mathrm{ind}(b)$$

Let $\mathcal{M}(a, b) = W^u(a) \cap W^s(b)$. Equivalently,

$$\mathcal{M}(a, b) = \left\{ p \in V : \lim_{s \to -\infty} \varphi^s(p) = a, \lim_{s \to \infty} \varphi^s(p) = b \right\} \tag{3}$$

Unbroken Flow Lines

Lemma

$\mathbb{R}$ acts on $\mathcal{M}(a,b)$ by

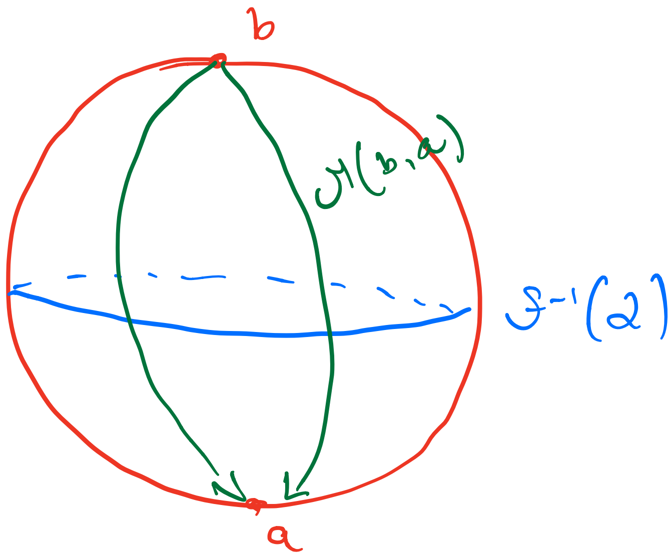
$$(s,p) \longmapsto \varphi^s(p)$$

The action is free if $a \neq b$.

Thus $\mathcal{L}(a,b) = \mathcal{M}(a,b)/\mathbb{R}$ is a manifold of dimension

$$\dim \mathcal{L}(a,b) = \operatorname{ind}(a) - \operatorname{ind}(b) - 1$$

Example



S^2 w/ natural height function

$\alpha \in (f(a), f(b)) \Rightarrow$

$\mathcal{L}(b,a) \cong \underbrace{\mathcal{M}(b,a) \cap f^{-1}(\alpha)}_{\text{equator}}.$

$\mathcal{L}(b,a)$ is the set of trajectories from b to a.

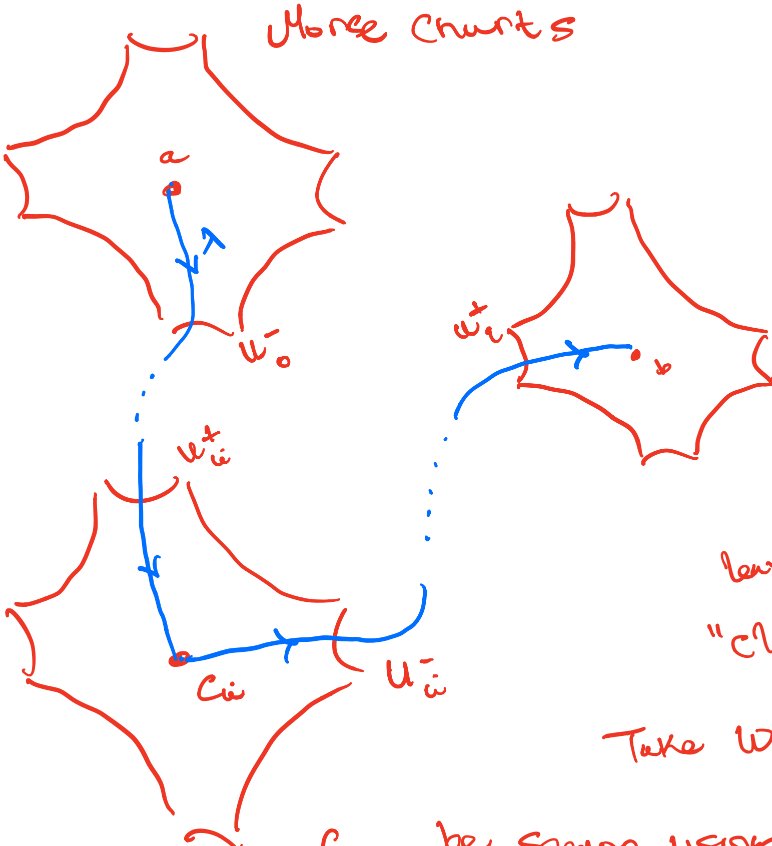
A Compactification of $\mathcal{L}(a, b)$

Let

$$\overline{\mathcal{L}}(a, b) = \bigcup_{c_i \in \mathrm{Crit}(f)} \mathcal{L}(a, c_1) \times \cdots \times \mathcal{L}(c_{q-1}, b). \tag{4}$$

By inducing a topology on $\overline{\mathcal{L}}(a, b)$ from the topology on V we may show that the space is a compactification of $\mathcal{L}(a, b)$.

A Topology on $\overline{\mathcal{L}}(a, b)$: The Idea



let $U^- = \bigcup_i U^-_{c_i}$, $U^+ = \bigcup_i U^+_{c_i}$

$\lambda = (\lambda_1, \dots, \lambda_q)$ is broken trajectory

Then

$$\mu \in W(\lambda, U^-, U^+)$$

if μ is broken trajectory that leaves $U^-_{c_i}$'s and enters $U^+_{c_i}$'s "close" to λ.

Take W's as basis of topology on $\overline{\mathcal{L}}(a, b)$.

Can be shown using sequences that $\overline{\mathcal{L}}(a, b)$ is compact.

Arbitrary Chain Complex

Let R be a commutative ring, $\{A_i\}_{i=1}^n$ a set of R-modules, and $A_\star$ the sequence

$$\cdots \to A_i \xrightarrow{\partial_i} A_{i-1} \to \cdots . \tag{5}$$

Definition

If $\partial_{i-1}\partial_i = 0$ then $A_\star$ is called a *chain complex*.

Note that this is equivalent to

$$\operatorname{im} \partial_i \subset \ker \partial_{i-1}. \tag{6}$$

Homology of a Chain Complex

Since $\ker \partial_{i-1} \subset \operatorname{im} \partial_i$ we have that the quotient group is well-defined:

Definition

The *i*-th *homology* group of the chain complex is the quotient group

$$H_i(A_\star) = \ker \partial_i / \operatorname{im} \partial_{i+1}. \tag{7}$$

The Morse Complex

Let V be a compact n-manifold with Morse function f and adapted pseudo-gradient X such that (f, X) is Morse-Smale.

Denote by $C_k(f)$ the free $\mathbb{Z}/2$-vector space generated by the critical points of f of index k.

Definition

The *Morse complex* is the sequence

$$\cdots \longrightarrow C_k(f) \xrightarrow{\partial_X} C_{k-1}(f) \longrightarrow \cdots$$

where ∂_X is defined on a critical point as $(w/\ a \in \mathrm{Crit}_k(f))$

$$\partial_X(a) = \sum_{c \in \mathrm{Crit}_{k-1}(f)} n_X(a,c)c \quad w/\ n_X(a,c) := |\mathcal{L}(a,b)| \bmod 2$$

The Morse Complex is a Complex

Let $a, b \in \mathrm{Crit}(f)$.

1. If $\mathrm{ind}(a) = \mathrm{ind}(b) + 1$ then

$\bar{\mathcal{L}}(a,b) = \mathcal{L}(a,b)$ is a compact 0-manifold $\Rightarrow$ finite set of points.

Hence $n_X(a,b)$ is well-defined

2. If $\mathrm{ind}(a) = \mathrm{ind}(b) + 2$ then (let $\mathrm{ind}(a) = k+1$, $\mathrm{ind}(b) = k-1$)

$\bar{\mathcal{L}}(a,b)$ is a compact 1-manifold. Can be shown

$$\partial \bar{\mathcal{L}}(a,b) = \coprod_{c \in \mathrm{Crit}_k(f)} \mathcal{L}(a,c) \times \mathcal{L}(c,b).$$

But compact 1-manifolds have an even number of boundary points $\Rightarrow$

$$|\partial \bar{\mathcal{L}}(a,b)| \equiv 0 \bmod 2$$

The Morse Complex is a Complex Cont.

Hence,

$$\partial_X \circ \partial_X (a) = \sum_{c \in Crit_k(f)} n_X(a,c)\, c$$

$$= \sum_{c \in Crit_k(f)} n_X(a,c) \left(\sum_{b \in Crit_{k-1}(f)} n_X(c,b)\, b \right)$$

$$= \sum_{b \in Crit_{k-1}(f)} \Big(\underbrace{\sum_{c \in Crit_k(f)} n_X(a,c)\, n_X(c,b)}_{= |\partial \overline{\mathcal{L}}(a,b)| \bmod 2 = 0} \Big) b$$

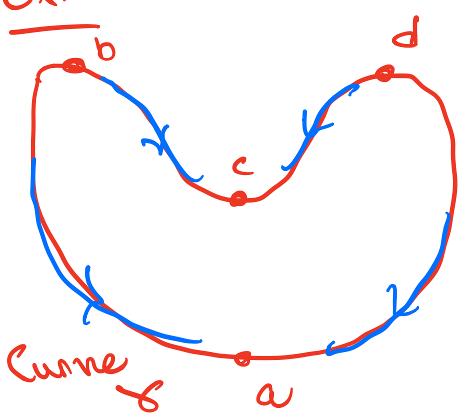
$$= 0.$$

So $\partial_X \circ \partial_X = 0 \Rightarrow$ Morse Complex is a complex.

Some Results

1. Morse homology is independent of the Morse-Smale pair (f, X)
2. Morse homology is isomorphic to singular homology
3. Morse homology is a topological invariant

Ex:



f = natural height

$$C_0(f) = \mathbb{Z}/_2 \oplus \mathbb{Z}/_2$$

$$C_1(f) = \mathbb{Z}/_2 \oplus \mathbb{Z}/_2$$

$$\partial_X(b) = a + c$$

$$\partial_X(d) = a + c$$

$$\Downarrow$$

$$H_0(f) = \mathbb{Z}/_2, \quad H_1(f) = \mathbb{Z}/2$$

References

[1] M. Audin and M. Damian. *Morse Theory and Floer Homology*. London: Springer-Verlag, 2014.

Acknowledgements

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