Some Generalizations of the Maslov Index

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Introduction

This is a brief overview of the material needed to define a version of the Maslov index. This will be used in following presentations. The following can be found in [1], [2] and is followed closely. The author makes no claims of originality.

Outline

Smooth Vector Bundles

Let M be a smooth n-manifold.

Definition

A **(real, smooth) vector bundle of rank k over M** is a smooth manifold E with a smooth map $E \xrightarrow{\pi} M$ such that:

1. All fibers E_p are real k-vector spaces,
2. For $p \in M$ there exists a neighborhood U and a diffeomorphism $\pi^{-1}(U) \xrightarrow{\Phi} U \times \mathbb{R}^k$ such that: ← called local trivializations
   - $\pi_U \circ \Phi = \pi$,
   - For each $q \in U$ the restriction Φ to E_q is a vector space isomorphism $E_q \to \{q\} \times \mathbb{R}^k \cong \mathbb{R}^k$.

π can be shown to be a smooth submersion

Think of the tangent or cotangent bundles of M!

Symplectic Vector Bundle

Let $E \xrightarrow{\pi} M$ be a vector bundle.

Definition

E is a **symplectic vector bundle** if there exists a family of symplectic forms

$$E_p \times E_p \xrightarrow{\omega_p} \mathbb{R} \tag{1}$$

on each fiber that varies smoothly in p.

These fit together to give an $\omega \in \Gamma(E^* \wedge E^*)$ that is non-degenerate.

Again, think of the tangent bundle of a symplectic manifold M!

Vector Bundle Constructions

Subbundles:

If $E \xrightarrow{\pi} M$, $S \xrightarrow{\pi'} M$ are vector bundles then

S is a subbundle of E if:

i) $S \hookrightarrow E$ embedded submanifold

ii) $S_p = S \cap E_p$ is the vector space structure

iii) $\pi' = \pi|_S$

Lagrangian Subbundles:

$E \longrightarrow M$ symplectic bundle & $S \longrightarrow M$ subbundle then

S is Lagrangian if:

$S_p \subset E_p$ Lagrangian $\forall p$.

Vector Bundle Constructions Cont.

Pullback Bundles:

$E \xrightarrow{\pi_M} M$ vector bundle & $N \xrightarrow{f} M$ smooth.

The pullback bundle $f^*E \xrightarrow{\pi_N} N$ is

$$f^*E = \{(n, e) \in N \times E : f(n) = \pi(e)\}.$$

If $f^*E \xrightarrow{\pi_E} E$ is projection then f^*E is s.t.

$$\begin{array}{ccc} f^*E & \xrightarrow{\pi_E} & E \\ \pi_N \downarrow & \circlearrowright & \downarrow \pi_M \\ N & \xrightarrow{f} & M \end{array}$$

Note: f^*E is universal w.r.t. this property

A Quick Lemma

Lemma

A symplectic vector bundle E over a compact oriented 2-manifold Σ with non-empty boundary $\partial\Sigma$ has a symplectic trivialization.

What is meant by a symplectic trivialization?

A global trivialization s.t. the induced isomorphisms

$$E_p \cong \mathbb{R}^{2n}$$

are actually linear symplectomorphisms

$$(E_p, \omega_p) \cong (\mathbb{R}^{2n}, \omega_0).$$

Maslov Index

Let $(\mathbb{R}^{2n}, \omega_0)$ be the standard symplectic space. Consider the Lagrangian Grassmanian:

$$\Lambda(n) = \{V : V \subset \mathbb{R}^{2n} \text{ is Lagrangian}\}. \tag{2}$$

Consider $\mathbb{C}^n \cong \mathbb{R}^{2n}$ under the standard identification. It can be shown any $V \in \Lambda(n)$ can be written as $A \cdot \mathbb{R}^n$ for $A \in U(n)$. Clearly $A \cdot \mathbb{R}^n = \mathbb{R}^n$ if and only if $A \in O(n)$. Hence

$$\Lambda(n) \cong U(n)/O(n). \tag{3}$$

Definition

For a loop $S^1 \xrightarrow{\gamma} \Lambda(n)$ we define the **Maslov index** as

$$\mu(\gamma) = \deg[(\det)^2 \circ \gamma]. \tag{4}$$

Totally Real Subspaces

A real subspace $V \subset \mathbb{C}^n$ is totally real if $V \cap iV = \{0\}$ and $\dim_{\mathbb{R}} V = n$. Let $\mathcal{R}(n)$ be the set of totally real subspaces.

It can be shown any such V can be written $V = A \cdot \mathbb{R}^n$ for some $A \in GL(n, \mathbb{C})$. Also $A_1\mathbb{R}^n = A_2\mathbb{R}^n$ if and only if $A_2^{-1}A_1 \in GL(n, \mathbb{R})$. Hence

$$\mathcal{R}(n) \cong GL(n, \mathbb{C})/GL(n, \mathbb{R}). \tag{5}$$

Generalizing The Maslov Index

Lemma

Let

$$\tilde{\mathcal{R}}(n) = \{D \in GL(n, \mathbb{C}) : D\overline{D} = I_n\}. \tag{6}$$

Then

$$\mathcal{R}(n) \xrightarrow{B} \tilde{\mathcal{R}}(n) \ : \ A \cdot \mathbb{R}^n \mapsto A^{-1}\overline{A} \tag{7}$$

is a diffeomorphism with respect to the standard smooth structures.

Corollary

Let $\tilde{\Lambda}(n) = B(\Lambda(n))$*, or equivalently,*

$$\tilde{\Lambda}(n) = \{D \in U(n) : D = D^t\}. \tag{8}$$

Then $B|_{\Lambda(n)}$ *is a diffeomorphism.*

Generalizing The Maslov Index Cont.

This is a generalization of the Maslov index to loops through totally real subspaces:

Definition

Let $S^1 \xrightarrow{\gamma} \mathcal{R}(n)$ be a loop. The **generalized Maslov index** $\mu(\gamma)$ is the winding number of

$$\det \circ B \circ \gamma : S^1 \to \mathbb{C} - \{0\} \tag{9}$$

Symplectic Bundle Pairs

Let Σ be a oriented compact surface with boundary $\partial\Sigma$ and h the number of ~~connection~~ connected components of $\partial\Sigma$.

Definition

A **symplectic bundle pair** is a pair $(\mathcal{V}, \lambda)$ over $(\Sigma, \partial\Sigma)$ where $\mathcal{V} \to \Sigma$ is a symplectic bundle and $\lambda \to \partial\Sigma$ is a Lagrangian subbundle of $\mathcal{V}|_{\partial\Sigma}$.

Fix a trivialization $\mathcal{V} \xrightarrow{\Psi} \Sigma \times (\mathbb{R}^{2n}, \omega_0)$. Then the restriction $\Psi(\lambda|_{\partial_i\Sigma})$ gives a loop

$$S^1 \xrightarrow{\gamma^i_{\Psi,\lambda}} \Lambda(n). \tag{10}$$

How do we get the loop?

Since Σ compact $\Rightarrow \partial\Sigma$ is compact.

Since $\partial_i \Sigma$ is connected component $\Rightarrow \partial_i\Sigma \subset \partial\Sigma$ closed $\Rightarrow$ $\partial_i \Sigma$ is compact.

So $\partial_i \Sigma$ is compact 1-manifold w/o boundary. Hence diffeomorphic to a circle. Let

$$S^1 \xrightarrow{\;\partial\;} \partial_i \Sigma$$

a parametrization. Then define

$$\gamma^i_{\varphi,\lambda}(t) = \varphi\left(\lambda|_{\partial(t)}\right).$$

Maslov Index of $(\mathcal{V}, \lambda)$

Let $\mu(\Psi, \partial_i \Sigma) = \mu(\gamma^i_{\Psi,\lambda})$.

Definition

The **Maslov index of the symplectic bundle pair** $(\mathcal{V}, \lambda)$ is

$$\mu(\mathcal{V}, \lambda) = \sum_{i=1}^{h} \mu(\Psi, \partial_i \Sigma). \tag{11}$$

See in [2] for proof that $\mu(\mathcal{V}, \lambda)$ is independent of trivialization Ψ.

Maslov Index of a Smooth Map $(\Sigma, \partial\Sigma) \xrightarrow{f} (M, L)$

Suppose we have a smooth map $(\Sigma, \partial\Sigma) \xrightarrow{f} (M, L)$ such that M is symplectic and $L \subset M$ is Lagrangian. We now have a symplectic bundle pair

$$(f^* TM, f^*|_{\partial\Sigma} TL) \tag{12}$$

associated to $(\Sigma, \partial\Sigma)$.

Definition

The **Maslov index of** f is

$$\mu_L(f) = \mu(f^* TM, f^*|_{\partial\Sigma} TL). \tag{13}$$

We note $\mu_L(f)$ is invariant under homotopy of f.

References

[1] D. McDuff and D. Salamon. *Introduction to Symplectic Topology*. New York: Oxford University Press, 1995.

[2] K. Fukaya [et al.] *Lagrangian Intersection Floer Theory*. AMS/IP Studies in Advanced Mathematics, 2009.

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