# Intuitive Intro to Floer Cohomology

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# Introduction

This is an intuitive introduction to Lagrangian intersection Floer cohomology. The following can be found in [1], [2] and is followed closely. The author makes no claims of originality.

# Outline

Let $(L_0, L_1)$ be a pair of compact and connected Lagrangian submanifolds of $(M, \omega)$. We define the functional

$$\tilde{\Omega}(L_0, L_1; \ell_0) \xrightarrow{\mathcal{A}} \mathbb{R} \tag{1}$$

$$\mathcal{A}[\ell, w] = \int w^* \omega$$

→ the critical points will be intersection points of $(L_0, L_1)$

→ the gradient flow lines will be strips connecting the intersection points

# Relation to Action 1-form

Let $\tilde{\Omega}(L_0, L_1; \ell_0) \xrightarrow{\pi} \Omega(L_0, L_1; \ell_0)$ be the $\Gamma$-covering projection. Then

$$dA = -\pi^* \alpha. \tag{2}$$

In particular, this shows that

the critical points $Cr(L_0, L_1; \ell_0)$ will be

$\{ [\ell_p, w] \in \Omega(L_0, L_1; \ell_0) :$ $\ell_p$ is constant path & $p \in L_0 \cap L_1 \}$

# $L^2$ Metric and Gradient Equation

Let $\{J_t\}_{t=0}^{1}$ be a family of almost complex structures on $M$ tamed by $\omega$. Define the metric on $\Omega(L_0, L_1; \ell_0)$ by

$$\langle \xi_1, \xi_2 \rangle_{J_t} = \int_0^1 \omega(\xi_1(t), J_t \xi_2(t))dt. \tag{3}$$

The gradient equation is then

$$(*) \begin{cases} \dfrac{du}{dt} + J_t \dfrac{du}{dt} = 0 \\ u(\tau, 0) \in L_0, \quad u(\tau, 1) \in L_1 \end{cases} \quad \text{w/ } \mathbb{R} \times [0,1] \xrightarrow{u} M$$

# Bounded Solutions

Since $\mathbb{R} \times [0,1]$ is not compact, we need a "decay" condition on $u$ satisfying $(*)$.

Let $\tilde{\mathcal{M}}^{reg} = \left\{ u : u \text{ satisfies } (*) \text{ and } \int_{\mathbb{R}\times[0,1]} u^* \omega < \infty \right\}$

Can then show if $u \in \tilde{\mathcal{M}}^{reg}$ and $L_0 \pitchfork L_1$, $\exists!$ $p, q \in L_0 \cap L_1$ s.t.

asymptotic condition

$$\lim_{\tau \to -\infty} u(\tau, \cdot) = c_p, \quad \lim_{\tau \to \infty} u(\tau, \cdot) = c_q.$$

Note: There is a natural $\mathbb{R}$-action on $\tau$. Let

$$\mathcal{M}^{reg} = \tilde{\mathcal{M}}^{reg} / \mathbb{R}$$

# Floer Cohomology

We can further decompose $\mathcal{M}^{reg}$ up to homotopy.

Let $\pi_2(p,q)$ be set of homotopy classes of $[0,1]^2 \xrightarrow{u} M$ s.t.

$$u(0,t) = p \qquad u(s,0) \in L_0$$
$$u(1,t) = q \qquad u(s,1) \in L_1 \ .$$

Let $\tilde{\mathcal{M}}^{reg}(p,q;B)$ be the set of $u$ satisfying $(*)$, asymptotic condition, and $[u] = B \in \pi_2(p,q)$. Write

$$\mathcal{M}^{reg}(p,q;[u]) = \tilde{\mathcal{M}}^{reg}(p,q;[u]) / \mathbb{R}$$

as before.

Finally, we need a way to index the points. We use the Maslov-Viterbo index $\mu(p,q;[u])$. See [2].

# Floer Cohomology Cont.

Let $(L_0, L_1)$ intersect transversally. Define

$$\mathrm{Int}(L_0, L_1; \ell_0) = \{ p \in L_0 \cap L_1 : \ell_p \in \Omega(L_0, L_1; \ell_0) \}$$

$$CF_{\mathbb{Z}_2}(L_0, L_1; \ell_0) = \text{free } \mathbb{Z}_2\text{-vector space generated by } \mathrm{Int}(L_0, L_1; \ell_0).$$

Then, assuming $\{J_t\}$ & $\mathcal{M}^{reg}(p, q; [u])\}$ satisfy some nice conditions, we have for $p \in \mathrm{Int}(L_0, L_1; \ell_0)$

$$\partial(p) := \sum_{\substack{q \in L_0 \cap L_1 \\ [u] \in \pi_2(p,q) \\ \mu(p,q;[u]) = 1}} \#_{\mathbb{Z}_2} \mathcal{M}^{reg}(p, q; [u]) \, q$$

satisfies $\partial^2 = 0$. When defined, Floer cohomology is invariant under Hamiltonian isotopy.

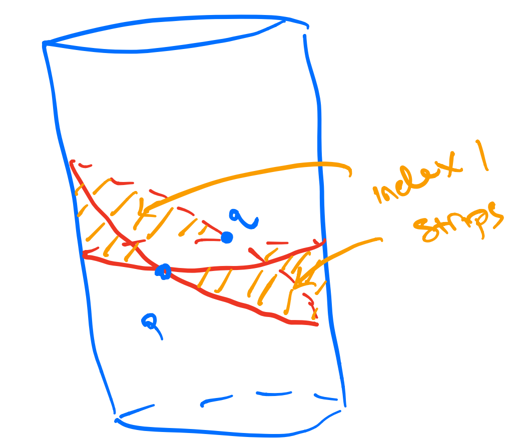
# Floer Cohomology Cont.

For example, Floer showed for $L \pitchfork \varphi_1^H(L)$ w/ $\varphi_1^H(t)$ a Hamiltonian diffeomorphism & if $\pi_2(M, L) = \{e\}$ then $\exists \{J_t\}$ s.t. we can define Floer cohomology.

In fact, under these conditions, it is isomorphic to the Morse Homology of $L$.

Ex:

$M = T^*S^1$




Hence $\partial(p) = \partial(q) = 0 \mod 2$.

$\Rightarrow HF = \mathbb{Z}_2$ which is the Morse homology of the circle!

# References


[1] D. McDuff and D. Salamon. *Introduction to Symplectic Topology*. New York: Oxford University Press, 1995.

[2] K. Fukaya [et al.] *Lagrangian Intersection Floer Theory*. AMS/IP Studies in Advanced Mathematics, 2009.

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