Exact Lagrangian Fillings of Legendrian Surfaces

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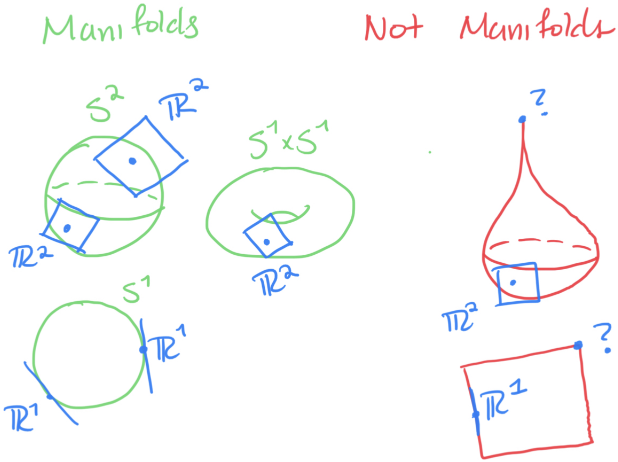
Overview

1. Contact Manifolds & Legendrian Submanifolds
2. Exact Lagrangian Fillings
3. What is Homology?
4. Filling $(\mathbb{S}^1)^n$

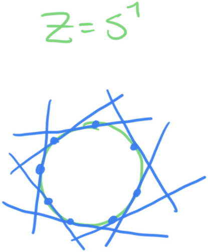
What is a Manifold?

A real manifold is any space that locally looks like $\mathbb{R}^n$. Similarly, a complex manifold is any space that locally looks like $\mathbb{C}^n$.



What is a Tangent bundle?

Let Z be an d-dimensional real manifold: so 'zoomed in' at every point of Z, we have a copy of $\mathbb{R}^d$. The tangent bundle TZ is given by 'gluing' all of these copies of $\mathbb{R}^d$ together.



$$TZ = \bigcup_{p \in Z} \{p\} \times T_p Z$$

Contact Manifolds

Take Z^{2n+1} a smooth odd-dimensional manifold and TZ its tangent bundle.

Definition (Technical)

A **contact structure** A contact structure on Z is a maximally non-integrable hyperplane field $\xi = \ker \alpha \subset TZ$, where α is a differential $1-$form.

A contact manifold is a pair (Z, ξ)

So, what does this even mean?

Example: Standard Contact form on $\mathbb{R}^3$

For example, take $M = \mathbb{R}^3$ (ie, $n = 1$) and $\alpha = dz - ydx$.

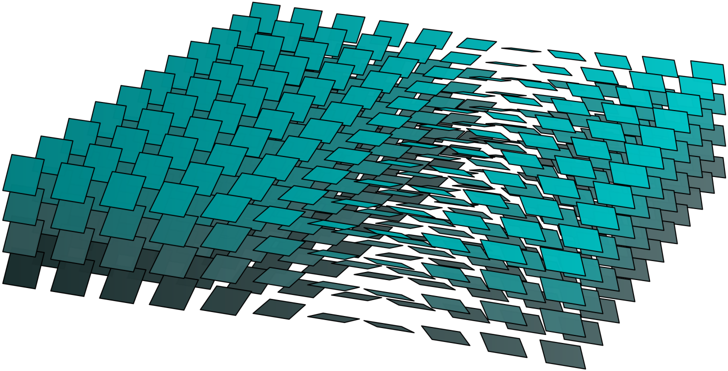


Figure: The standard contact structure on $\mathbb{R}^3$, from Wikimedia Commons.

Legendrian Submanifolds

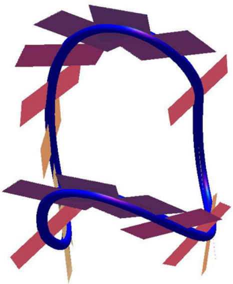
Definition

A **Legendrian Submanifold** $\Lambda \subseteq M$ of a smooth $2n+1$ dimensional contact manifold satisfies the following:

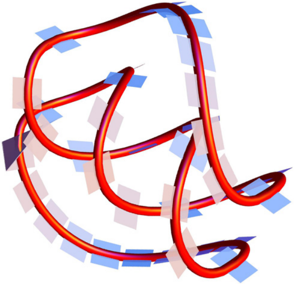
1. Λ is tangent to ξ: $T\Lambda_p \subseteq \xi_p$ for all $p \in \Lambda$ *(isotropic)*
2. $\dim \Lambda = n$ *(maximal)*

In other words, a Legendrian submanifold is a submanifold that is everywhere tangent to the hyperplane field $\xi = \ker \alpha$.

Ex. Legendrian Submanifolds of $\mathbb{R}^3$



(a) Legendrian Unknot



(b) Legendrian Right-handed trefoil

Figure: Legendrian Submanifolds of $\mathbb{R}^3$ endowed with the standard contact form $\alpha = dz - ydx$. From the Duke Gallary of Legendrian knots.

Exact Lagrangian Fillings

Let Λ_+ and Λ_- be Lagrangian submanifolds of $(\mathbb{R}^3, \ker \alpha)$.

Definition

An **Exact Lagrangian Cobordism** $\Sigma \subset (\mathbb{R} \times \mathbb{R}^3, d(e^t\alpha))$ is a 2-dimensional surface such that there exists a $T > 0$ such that:

1. Σ is cylindrical over Λ_+ in the interval (T, ∞):

$$\Sigma \cap (T, \infty) \times \mathbb{R}^3) = (T, \infty) \times \Lambda_+,$$

2. Σ is cylindrical over Λ_- in the interval (-T,-∞):

$$\Sigma \cap (-\infty, T) \times \mathbb{R}^3) = (-\infty, T) \times \Lambda_-,$$

3. Σ is compact in in $[T, T] \times \mathbb{R}^3$,
4. $e^t\alpha|_{T\Sigma} = df$ for some function $f : \Sigma \to \mathbb{R}$.

Exact Lagrangian Fillings

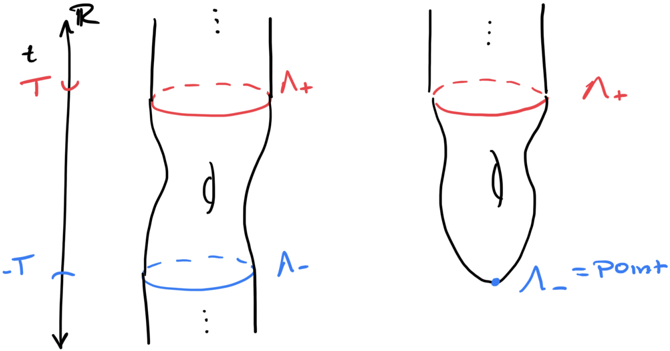


Figure: An Exact Lagrangian Cobordism and an Exact Lagrangian Filling.

Definition

If we can take $\Lambda_- = \emptyset$, then Σ is an **Exact Lagrangian Filling** of Λ_+.

Example: $\Lambda = S^1 \subset S^3$

To find an exact Lagrangian filling of S^1, we can consider the disk D^2 embedded by the following 'diagonal' map:

$$u : D^2 \to Z$$

$$z \mapsto (z, -z)$$

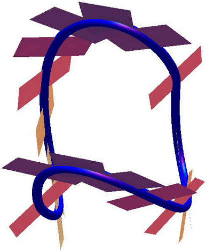
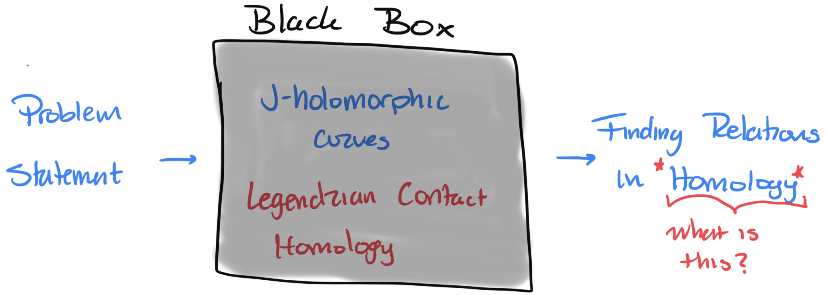


Figure: $S^1 \subset \mathbb{R}^3$

The question

Take Z^{2n+1} a circle-bundle over $\Lambda = (S^1)^n$. For which n do there not exist some exact Lagrangian filling L?



What is Homology

Homology is a way of reframing questions in topology as questions in algebra. *Disclaimer: the information presented in this section is not neccessarily technically accurate; it is merely to give intuition for what homology is.*

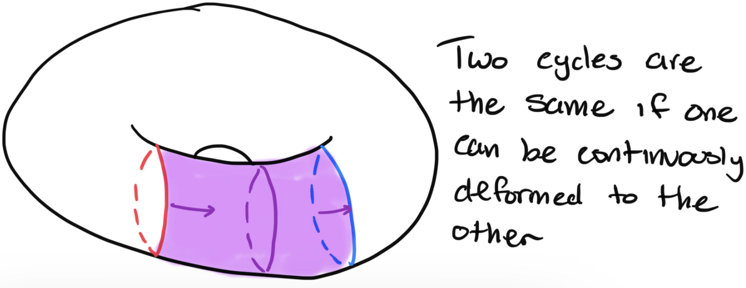
Topologists only care about holes. So we want a way of counting how many n-dimensional holes a space has. To do this, we'll consider cycles and boundaries.

Cycles & Boundaries

1-cycles in the torus

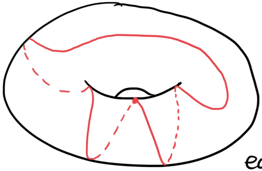
If a cycle bounds a solid disk, it's a boundary

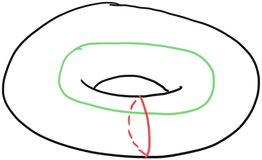
Homology = {Cycles that aren't boundaries}/{cycles that are the same}



To compute the homology of a space, consider all possible cycles in it. Then, check if that cycle is the boundary of a disk. If it isn't, then we've detected a hole!

Winding number





Any 1-cycle that isn't a boundary can, up to continuous deformation, be completely characterized by how many times it loops around the green curve and how many times it loops around the red curve.
Thus $H_1(S^1 \times S^1) = \mathbb{Z} \times \mathbb{Z}$.

My Work

Take $\iota : \Lambda \to L$ to be the inclusion map, and consider its pullback in homology $\iota_* : H(\Lambda) \to H(L)$.

Lemma (Adapted from Lem. 3.2.4 of WW22)

The image of $H_1(\Lambda) \oplus H_{n-1}(\Lambda)$ in $H_1(L) \oplus H_{n-1}(L)$ is half-dimensional.

Thus, we can disprove the existence of our filling L by considering the number of homology classes in $\mathrm{im}(H_1(\Lambda) \oplus H_{n-1}(\Lambda))$. If this number is too high, L can't exist!

Conjecture (to be proved (hopefully) soon)

For $n > 2$, $(S^1)^n$ admits no exact Lagrangian filling.

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