Exact Lagrangian Fillings of Legendrian Surfaces

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Overview

1. Contact Manifolds & Legendrian Submanifolds
2. Exact Lagrangian Fillings
3. What is Homology?
4. Filling $(\mathbb{S}^1)^n$
What is a Manifold?

A real manifold is any space that locally looks like $\mathbb{R}^n$. Similarly, a complex manifold is any space that locally looks like $\mathbb{C}^n$. 

Manifolds

$S^2$, $\mathbb{R}^2$, $S^1 \times S^1$, $\mathbb{R}^1 \times \mathbb{R}^1$

Not Manifolds

$\mathbb{R}^1$, $\mathbb{R}^2$, $\mathbb{C}^2$
What is a Tangent bundle?

Let $Z$ be an $d$-dimensional real manifold: so ‘zoomed in’ at every point of $Z$, we have a copy of $\mathbb{R}^d$. The tangent bundle $TZ$ is given by ‘gluing’ all of these copies of $\mathbb{R}^d$ together.
Contact Manifolds

Take $Z^{2n+1}$ a smooth odd-dimensional manifold and $TZ$ its tangent bundle.

Definition (Technical)

A contact structure A contact structure on $Z$ is a maximally non-integrable hyperplane field $\xi = \ker \alpha \subset TZ$, where $\alpha$ is a differential 1–form.

A contact manifold is a pair $(Z, \xi)$

So, what does this even mean?
Example: Standard Contact form on $\mathbb{R}^3$

For example, take $M = \mathbb{R}^3$ (ie, $n = 1$) and $\alpha = dz - ydx$.

Figure: The standard contact structure on $\mathbb{R}^3$, from Wikimedia Commons.
Legendrian Submanifolds

Definition

A **Legendrian Submanifold** $\Lambda \subseteq M$ of a smooth $2n + 1$ dimensional contact manifold satisfies the following:

1. $\Lambda$ is tangent to $\xi$: $T\Lambda_p \subseteq \xi_p$ for all $p \in \Lambda$ (*isotropic*)
2. $\dim \Lambda = n$ (*maximal*)

In other words, a Legendrian submanifold is a submanifold that is everywhere tangent to the hyperplane field $\xi = \ker \alpha$. 
Ex. Legendrian Submanifolds of $\mathbb{R}^3$

(a) Legendrian Unknot

(b) Legendrian Right-handed trefoil

Figure: Legendrian Submanifolds of $\mathbb{R}^3$ endowed with the standard contact form $\alpha = dz - ydx$. From the Duke Gallery of Legendrian knots.
Exact Lagrangian Fillings

Let $\Lambda_+$ and $\Lambda_-$ be Lagrangian submanifolds of $(\mathbb{R}^3, \ker \alpha)$.

**Definition**

An **Exact Lagrangian Cobordism** $\Sigma \subset (\mathbb{R} \times \mathbb{R}^3, d(e^t \alpha))$ is a 2-dimensional surface such that there exists a $T > 0$ such that:

1. $\Sigma$ is cylindrical over $\Lambda_+$ in the interval $(T, \infty)$:
   \[ \Sigma \cap (T, \infty) \times \mathbb{R}^3 = (T, \infty) \times \Lambda_+, \]

2. $\Sigma$ is cylindrical over $\Lambda_-$ in the interval $(-\infty, -T)$:
   \[ \Sigma \cap (-\infty, T) \times \mathbb{R}^3 = (-\infty, T) \times \Lambda_-, \]

3. $\Sigma$ is compact in $[T, T] \times \mathbb{R}^3$,

4. $e^t \alpha |_{T\Sigma} = df$ for some function $f : \Sigma \to \mathbb{R}$. 


Definition

If we can take $\Lambda_- = \emptyset$, then $\Sigma$ is an Exact Lagrangian Filling of $\Lambda_+$. 

Figure: An Exact Lagrangian Cobordism and an Exact Lagrangian Filling.
Example: $\Lambda = S^1 \subset S^3$

To find an exact Lagrangian filling of $S^1$, we can consider the disk $D^2$ embedded by the following ‘diagonal’ map:

$$u : D^2 \to \mathbb{Z}$$

$$z \mapsto (z, -z)$$

Figure: $S^1 \subset \mathbb{R}^3$
The question

Take $Z^{2n+1}$ a circle-bundle over $\Lambda = (S^1)^n$. For which $n$ do there not exist some exact Lagrangian filling $L$?
What is Homology

Homology is a way of reframing questions in topology as questions in algebra. *Disclaimer: the information presented in this section is not necessarily technically accurate; it is merely to give intuition for what homology is.*

Topologists only care about holes. So we want a way of counting how many $n$-dimensional holes a space has. To do this, we’ll consider cycles and boundaries.
Cycles & Boundaries

1-cycles in the torus

If a cycle bounds a solid disk, it's a boundary
Homology = \{\text{Cycles that aren’t boundaries}\} / \{\text{cycles that are the same}\}

To compute the homology of a space, consider all possible cycles in it. Then, check if that cycle is the boundary of a disk. If it isn’t, then we’ve detected a hole!
Winding number

Any 1-cycle that isn’t a boundary can, up to continuous deformation, be completely characterized by how many times it loops around the green curve and how many times it loops around the red curve. Thus $H_1(S^1 \times S^1) = \mathbb{Z} \times \mathbb{Z}$.
Take \( \iota : \Lambda \to L \) to be the inclusion map, and consider its pullback in homology \( \iota^* : H(\Lambda) \to H(L) \).

**Lemma (Adapted from Lem. 3.2.4 of WW22)**

The image of \( H_1(\Lambda) \oplus H_{n-1}(\Lambda) \) in \( H_1(L) \oplus H_{n-1}(L) \) is half-dimensional.

Thus, we can disprove the existence of our filling \( L \) by considering the number of homology classes in \( \text{im}(H_1(\Lambda) \oplus H_{n-1}(\Lambda)) \). If this number is too high, \( L \) can’t exist!

**Conjecture (to be proved (hopefully) soon)**

For \( n > 2 \), \( (S^1)^n \) admits no exact Lagrangian filling.
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K. Blakley, S. Chanda, Y. Sun, and C. Woodward. “Augmentation varieties and disk potentials” *In preparation*