Project: Classifying Exact Lagrangian Fillings of Legendrian Surfaces

Prepared while participating in the 2023 DIMACS REU

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June 5, 2023

Overview

1. What is a Legendrian Surface?
2. What is an Exact Lagrangian Filling?

The following can be found in [1] and [2], which are followed closely. The author makes no claims of originality.

Contact Manifolds

Take M a smooth manifold and TM its tangent bundle.

Definition

A **hyperplane field** ξ is a codimension 1 subbundle of TM.

Locally, ξ can be described as $\ker \alpha$ where α is a differential 1-form.

Definition

A **contact structure** on a $2n+1$ dimensional manifold M is a maximally non-integrable hyperplane field.

In other words, no hyperplane $\Pi \subset M$ can be tangent to ξ along any open subset of Π.

Standard Contact form on $\mathbb{R}^3$

For example, take $M = \mathbb{R}^3$ (ie, $n = 1$) and $\alpha = dz - ydx$.

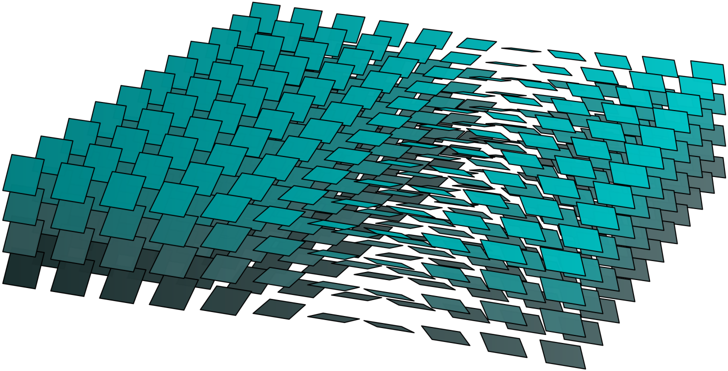


Figure: The standard contact structure on $\mathbb{R}^3$, from Wikimedia Commons.

Legendrian Submanifolds

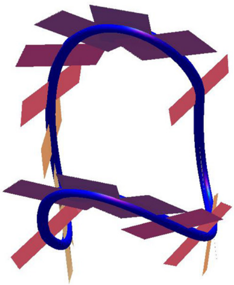
Definition

A **Legendrian Submanifold** $\Lambda \subseteq M$ of a smooth $2n+1$ dimensional contact manifold satisfies the following:

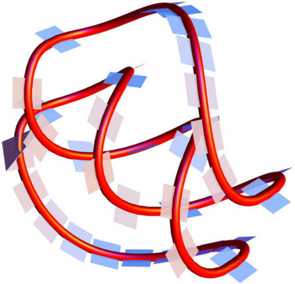
1. Λ is tangent to ξ: $T\Lambda_p \subseteq \xi_p$ for all $p \in \Lambda$ *(isotropic)*
2. $\dim \Lambda = n$ *(maximal)*

In other words, a Legendrian submanifold is a submanifold that is everywhere tangent to the hyperplane field $\xi = \ker \alpha$.

Ex. Legendrian Submanifolds of $\mathbb{R}^3$



(a) Legendrian Unknot



(b) Legendrian Right-handed trefoil

Figure: Legendrian Submanifolds of $\mathbb{R}^3$ endowed with the standard contact form $\alpha = dz - ydx$. From the Duke Gallary of Legendrian knots.

Exact Lagrangian Fillings

Let Λ_+ and Λ_- be Lagrangian submanifolds of $(\mathbb{R}^3, \ker \alpha)$.

Definition

An **Exact Lagrangian Cobordism** $\Sigma \subset (\mathbb{R} \times \mathbb{R}^3, d(e^t\alpha))$ is a 2-dimensional surface such that there exists a $T > 0$ such that:

1. Σ is cylindrical over Λ_+ in the interval (T, ∞):

$$\Sigma \cap (T, \infty) \times \mathbb{R}^3) = (T, \infty) \times \Lambda_+,$$

2. Σ is cylindrical over Λ_- in the interval (-T,-∞):

$$\Sigma \cap (-\infty, T) \times \mathbb{R}^3) = (-\infty, T) \times \Lambda_-,$$

3. Σ is compact in in $[T, T] \times \mathbb{R}^3$,
4. $e^t\alpha|_{T\Sigma} = df$ for some function $f : \Sigma \to \mathbb{R}$.

Exact Lagrangian Fillings

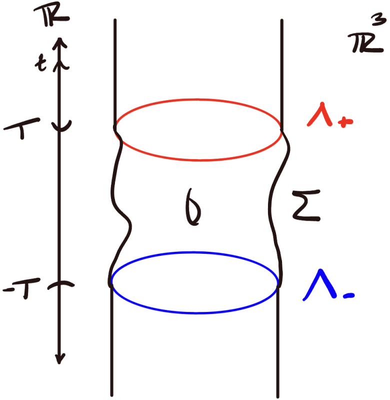


Figure: An Exact Lagrangian Cobordism.

Definition

If we can take $\Lambda_- = \emptyset$, then Σ is an **Exact Lagrangian Filling** of Λ_+.

The Project

The above examples are for $(\mathbb{R}^3, \ker \alpha)$. What if we take M to be a contact 5-manifold? Then Λ is a 2-dimensional surface, for instance Symplectic Toric Manifolds. Given some toric variety, can we classify the exact Lagrangian fillings?

Acknowledgements

1. I'd like to thank my mentor Christopher Woodward for working with me this summer.
2. Work supported by the Rutgers Department of Mathematics.

References

- [1] Yu Pan. Exact Lagrangian fillings of Legendrian (2, n) torus links. *Pacific J. Math.*, 289(2):417–441, 2017. arXiv:1607.03167, doi:10.2140/pjm.2017.289.417.
- [2] Contact Manifolds Encyclopedia of Mathematical Physics, eds. J.-P. Françoise, G.L. Naber and S.T. Tsou, Oxford: Elsevier, v. 1 2006631–636.