# Designing non-manipulable tournament rules

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# Motivation

- Consider the following tournament where all teams play against each other exactly once.

| Tournament Results | | | | |
|---|---|---|---|---|
| | A | B | C | D |
| A | | Draw | Win | Draw |
| B | Draw | | Loss | Win |
| C | Loss | Win | | Win |
| D | Draw | Loss | Loss | |

| Points table | |
|---|---|
| C | 6 |
| A | 5 |
| B | 4 |
| D | 1 |

- Suppose that the match between $A$ and $B$ was played as the last. Could they colluded so that one of them would won the tournament?

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- It is easy to see that they could.

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| A | | Win | Win | Draw |
| B | Loss | | Loss | Win |
| C | Loss | Win | | Win |
| D | Draw | Loss | Loss | |

| Points table | |
|---|---|
| A | 7 |
| C | 6 |
| B | 3 |
| D | 1 |

- We would like to come up with different rules to determine winners of tournaments that is "reasonable" and teams don't have any incentive to collude.

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**Theorem**

*There exists a rule that is (CC) and (2-SNM-1/3) and no better rule exists.*

# Our goals

- Generally, we consider the case when $k$ teams collude:
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- Our goal is to improve the lower bound on $\alpha$ (if possible), for some $k \geq 3$, and find a rule that achieve this lower bound.
- We will try to come up with other reasonable ways how to relax (CC) and ($k$-SNM) properties.