A graph game from extremal combinatorics

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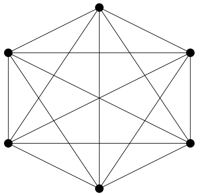
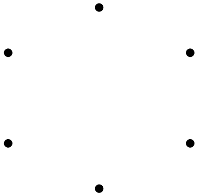
Notation

- Graph $G :=$ Ordered pair (V, E) of vertices and edges.
- $K_n :=$ Clique on n vertices. I.e. $K_n = \left(V, \binom{V}{2}\right)$.
- Independent set in G is a subset of vertices such that no two of them are connected by an edge.

Clique

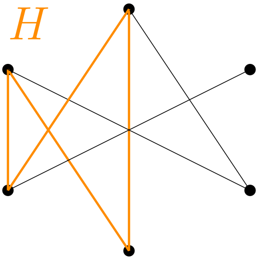


Independent set



Notation

- H is an induced subgraph of G if there exists a subset A of vertices of G, such that $G[A] = H$.



The game

For a given integers m and $k \geq 2$, and a fixed graph H consider a following game played on a graph G

Definition (Forcing game $\mathcal{F}(H, k, m)$)

We start with a graph G with m vertices and no edges. Then players A and B take turns, as follows:

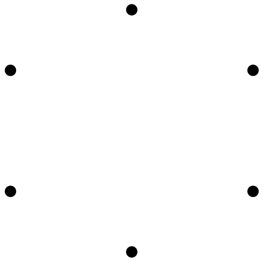
- Player A either selects an independent set S of size k in G or decides to stop the game.
- Player B modifies G by adding edges with both ends in S; he must add at least one edge, but may add more.

At the end of the game, Player A wins if G contains H as an induced subgraph; and Player B wins otherwise.

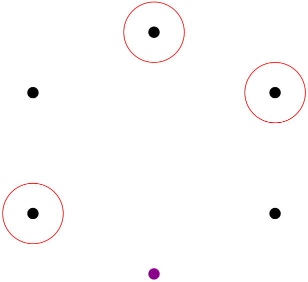
Example

- $k = 3$
- $H = K_3$ (triangle)
- $m = 6$

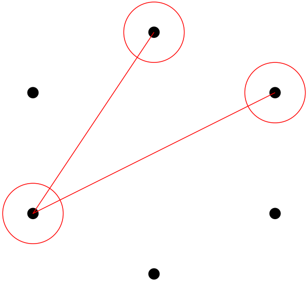
Example. Turn 1.



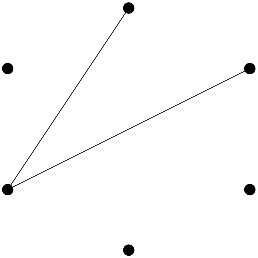
Example. Turn 1: Player A.



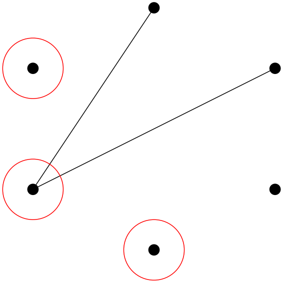
Example. Turn 1: Player B.



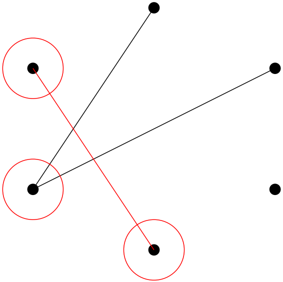
Example. Turn 2.



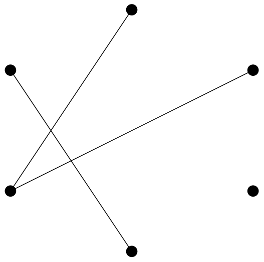
Example. Turn 2. Player A.



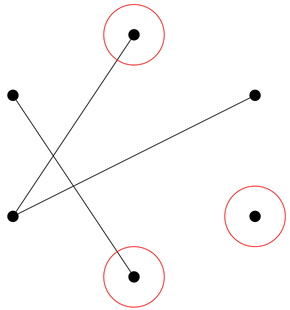
Example. Turn 2. Player B.



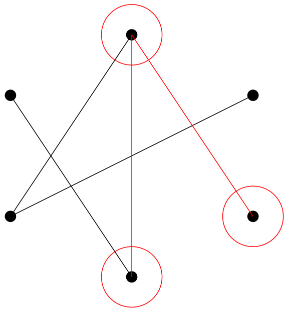
Example. Turn 3.



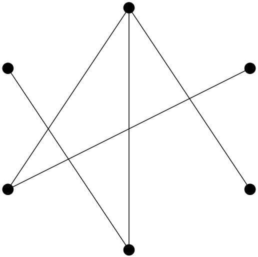
Example. Turn 3. Player *A*.



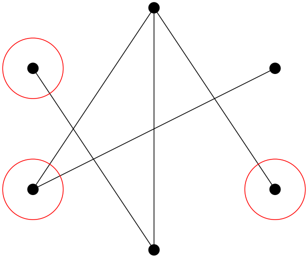
Example. Turn 3. Player B.



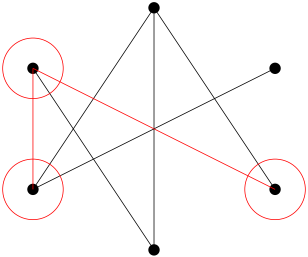
Example. Turn 4.



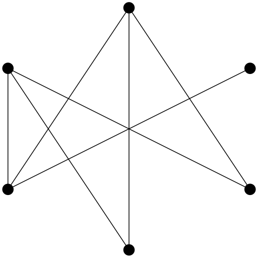
Example. Turn 4. Player A.



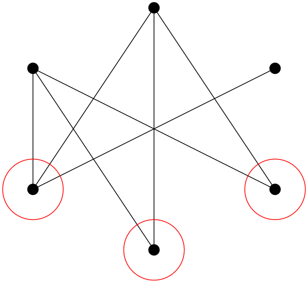
Example. Turn 4. Player B.



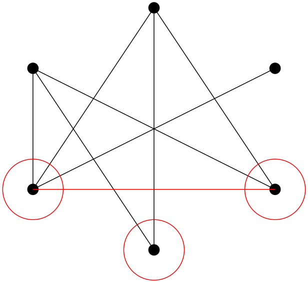
Example. Turn 5.



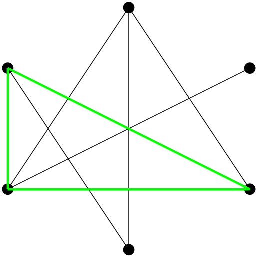
Example. Turn 5. Player *A*.



Example. Turn 5. Player *B*.



Example. Player A wins.



Additional definitions

Definition

Let $N(H, k)$ be the minimal m such that player A can always win the game $\mathcal{F}(H, k, m)$. (No matter how the player B plays).

Definition (Ramsey number)

Let $R(t, k)$ be the smallest n such that every graph on at least n vertices contains either K_t or an independent set on k vertices.

Known facts

- For every k and H, player A wins for every sufficiently large graph. (i.e. $N(H,k) < \infty$).
- $N(H,k)$ is bounded from above by some function double exponential in k.
- $N(K_t,k) = R(t,k)$ (i.e. it coincides with the Ramsey number). And thus it is bounded from below by some function exponential in k.

Our goal

- Find better bounds for $N(H, k)$.
- Explore other variants of the game. For instance when player A does not see the graph G and may chose to stop the game in his turn and:
  - Player A loses when he choses a set which is not independent and he wins if G contains induced H in the end.
  - Player A loses when he choses a set which is not independent and he wins if G contains induced H any time during the game.
  - Turn is skipped if player A choses a set which is not independent and he wins if G contains induced H in the end.
  - Turn is skipped if player A choses a set which is not independent and he wins if G contains induced H any time during the game.

For these variants it is not even known if player A can always win on sufficiently large graphs.