A Complete Generalization of Göllnitz's Big Theorem

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Abstract

In the spirit of Göllnitz's "big" partition theorem of 1967, we present a new mod 6 partition identity. Alladi, Andrews, and Gordon provided a refinement of Göllnitz's big theorem in 1995 via a "key identity" and the method of weighted words. Using this technique, two similar identities of Göllnitz type were discovered by Alladi (in 1999) and by Alladi and Andrews (in 2015). We finish the picture by presenting and proving the fourth and final possible mod 6 identity. Furthermore, we provide the complete generalization of these for all moduli n at least 6.

Preliminaries

Definition. A partition λ of a positive integer n is a non-increasing sequence of positive integers.
If $P(n)$ is the number of partitions of n, then its generating function

$$\sum_{n\geq 0} P(n)q^n = \prod_{i>0} \frac{1}{1-q^i}$$

Often, we use the q-Pochhammer symbol $(a)_n = \prod_{i=0}^{n-1}(1-aq^i)$.
So the generating function above may be written $\frac{1}{(q)_\infty}$

A partition identity equates number of partitions counted under certain restriction to another. For example, the number of partitions of an integer n into odd parts is the same as the number of partitions of n into distinct parts (Euler).

Theorem(Göllnitz 1967).
Let $P(n)$ denote the number of partitions of n into distinct parts $\equiv$ 2,4,5 (mod 6).
Let $S(n)$ denote the number of partitions of n where parts differ by ≥ 6 with equality only if a part $\equiv$ 2,4,5 (mod 6). In addition, 1 and 3 do not appear as parts.
Then $P(n) = S(n)$ for all positive integer n.

IdentityFinder

`MAPLE` package `IdentityFinder` of S.Kanade and M.Russell was successful in rediscovering existing identities and conjecturing new identities. We implemented new version of `IdentityFinder` in `C` and `JAVA`.

Examples

These are two other mod 6 identities

Theorem (Alladi 1995). $a = 1, b = 3, c = 5$.

Theorem (Alladi-Andrews 2015). $a = 1, b = 2, c = 4$.

a=1, b=2, c=4	a=1, b=2, c=5	a=1, b=2, c=6	a=1, b=3, c=5
a=1, b=3, c=7	a=1, b=4, c=6	a=1, b=4, c=7	a=1, b=5, c=7
a=2, b=3, c=4	a=2, b=3, c=6	a=2, b=4, c=5	a=2, b=4, c=7
a=2, b=5, c=6	a=2, b=6, c=7	a=3, b=4, c=5	a=3, b=4, c=6
a=3, b=5, c=7	a=3, b=6, c=7	a=4, b=5, c=6	a=4, b=6, c=7

Mod 8 Identities.

New Identity

Let $P(n)$ denote the number of partitions of n into distinct parts $\equiv$ 2,3,4 (mod 6).
Let $S(n)$ denote the number of partitions of n where parts differ by ≥ 6 with equality only if a part $\equiv$ 2,3,4 (mod 6). In addition, 1 does not appear as a part, yet 7+2 is allowed.
Then $P(n) = S(n)$ for all positive integer n.

Weighted Words

The method of weighted words build up on the *key identity* established by Alladi, Andrews, and Gordon.

$$\sum_{i,j,k} a^i b^j c^k \sum_{\substack{s=\alpha+\beta+\gamma+\delta+\epsilon+\phi \\ i=\alpha+\delta+\epsilon, j=\beta+\delta+\phi, k=\gamma+\epsilon+\phi}} \frac{q^{T_s+T_\delta+T_\epsilon+T_{\phi-1}}(1-q^\alpha(1-q^\phi))}{(q)_\alpha(q)_\beta(q)_\gamma(q)_\delta(q)_\epsilon(q)_\phi} = \sum_{i,j,k} \frac{a^i b^j c^k q^{T_i+T_j+T_k}}{(q)_i(q)_j(q)_k} = (-aq)_\infty(-bq)_\infty(-cq)_\infty$$

where $T_m = 1 + 2 + \cdots + m = \frac{m(m+1)}{2}$.
Given this key identity, we obtain our new identity with substitutions

$$\text{(dilation)} \quad q \mapsto q^6, \quad \text{and (translations)} \quad a \mapsto aq^{-4}, b \mapsto bq^{-3}, c \mapsto cq^{-2} \tag{1}$$

On the product side, it is rather immediate as we get $(-aq^2)_\infty(-bq^3)_\infty(-cq^4)_\infty$.

Let a, b, c be primary colors and ab, ac, bc be secondary colors. An integer n with color a is denoted a_n and similarly for other colors.
We order these colored integers by the following scheme:

$$\underline{ab}_1 < \underline{ac}_1 < \underline{bc}_1 < a_1 < b_1 < c_1 < ab_2 < ac_2 < bc_2 < a_2 < \cdots \tag{Scheme IV}$$

and order on each level is

$$ab < ac < bc < a < b < c$$
$$ab < ac < bc < a < b < c$$

Then each colored part translate reads

$$a_m \mapsto 6m-4, \quad b_m \mapsto 6m-3, \quad c_m \mapsto 6m-2$$
$$ab_m \mapsto 6m-7, \quad ac_m \mapsto 6m-6, \quad bc_m \mapsto 6m-5 \tag{2}$$

Then the sum-side partitions of our new identity are given by colored partitions of certain type:

Definition. A colored partition into parts in Scheme IV is of *Type-IV* if the distance between each part ≥ 1 with equality only if color of greater part is also greater, or adjacent parts are of the same primary color, but $bc_2 + a_1$ is allowed.

$$(7 + 6 + 5 + 4 + 3 + 2) \mapsto (40 + 33 + 26 + 19 + 12 + 5)$$

Generalization

The key property in the weighted words is that primary and secondary colors are distinct modulo 6. Moreover, partitions are given in terms of colored partitions, so with suitable dilations and translations, we obtain infinite family of identities of Göllnitz type.
In fact, what produces different colored partitions is the ordering on colors. There is no other way of ordering 6 primary and secondary colors different from the four mod 6 identities. Hence we reach the general theorem.

Theorem. Let $\{a < b < c\} \subset \{1, 2, \ldots, N\}$ be of Göllnitz-type, i.e. a, b, c and pairsums are distinct classes modulo N. Let $P(n)$ count the number of partitions of n into distinct parts congruent to a, b, c mod N.
Let $S(n)$ count the number of partitions λ satisfying the following conditions

- Parts are allowed if and only if congruent to a, b, c, ab, ac, bc mod N.
- $\lambda_i - \lambda_{i+1} \geq N$ with equality only if λ_i is congruent to a, b, c mod N.
- No parts are smaller than a, b, c, ab, ac, bc in each congruence class.
- $bc + a$ is always allowed.

Then $S(n) = P(n)$ for all n.

Equivalence of Identities

Let a, b, and c be three classes mod 6 of Göllnitz type. Quotient form of n is a unique expression as a multiple of 6, plus $a, b, c, a+b, a+c$, or $b+c$.
We define the multiplicity of an integer to be the number we multiply by 6 in quotient form. We define the a-multiplicity of an integer to be 1 if it is congruent to $a, a+b$, or $a+c$ and 0 otherwise. We likewise define b-multiplicity and c-multiplicity.
We define the multiplicity of a partition to be the sum of the multiplicities of the parts in the partition, and likewise the a-multiplicity, b-multiplicity, and c-multiplicity of a partition. We show the number of partitions with specified a, b, c-multiplicities and multiplicity (which determines the integer being partitioned) is the same for all four sumsides in the following way:
We construct ϕ_i as a map from the sum-side of Theorem 1 (S_1) to the sum-side of Theorem i (S_i) by writing the partitions counted by S_1 in quotient form and change all the 2s, 4s, and 5s to the corresponding values of a, b, and c (respectively) in S_i. Then, we look for the first i such that $\lambda_i - \lambda_{i+1} < 6$ and set $\lambda_i' = \lambda_{i+1} + 6$ and $\lambda_{i+1}' = \lambda_i - 6$. If we encounter $6+1$ (in ϕ_3)or $7+2$ (in ϕ_4), we leave these be to avoid forbidden parts.
For example, applying ϕ_4 to $(40, 33, 26, 20, 9, 2)$:

$$(6\cdot 6+4)+(6\cdot 4+[4+5])+(6\cdot 4+2)+(6\cdot 3+2)+(6\cdot 0+[4+5])+(6\cdot 0+2) \to$$

$$(6\cdot 6+3)+\overbrace{(6\cdot 4+[3+4])+(6\cdot 4+2)}+(6\cdot 3+2)+(6\cdot 0+[3+4])+(6\cdot 0+2) \to$$

$$(6\cdot 6+3)+(6\cdot 5+2)+\overbrace{(6\cdot 3+[3+4])+(6\cdot 3+2)}+(6\cdot 0+[3+4])+(6\cdot 0+2) \to$$

$$(6\cdot 6+3)+(6\cdot 5+2)+(6\cdot 4+2)+(6\cdot 2+[3+4])+(6\cdot 0+[3+4])+(6\cdot 0+2)$$

Which is the partition $(39, 32, 26, 19, 7, 2)$

Remarks

In fact, we applied the same argument to generalize a four-color identity of Alladi, Andrews and Berkovich.

Besides the method of weighted words, we also wrote a classical 2-parameter refinemenet proof and a combinatorial proof.

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