NOTES ON REPRESENTATION THEORY OF VERTEX ALGEBRAS

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1. Abstract

Last time, we showed the equivalence of a V-module structure on W and a representation of V on W in the setting of the canonical weak vertex algebra $\mathcal{E}(W)$. We will determine when a subalgebra of $\mathcal{E}(W)$ is indeed a vertex algebra. Moreover, focusing on the subalgebra $\mathcal{E}(W, d)$, we will determine vertex operator subalgebras. Then we will go over some construction theorems for vertex algebras and modules in a similar spirit.

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$a(x) \in \mathcal{E}(W)$, then $Y_W(a(x), x_0) = a(x_0)$.

3. Introduction

Previous week, Terence introduced the notion of weak vertex operators and weak vertex algebras. Moreover, Terence established that V-module structure on a vector space W is equivalent to giving a representation of V on W. However, our discussion was on the canonical weak vertex algebra $\iota_W(V) = (\mathcal{E}(W), Y_{\mathcal{E}}, 1_W)$, which usually is not a vertex algebra. To bring in vertex algebras in this picture, which is our interest, we introduce the notion of subalgebras of $\mathcal{E}(W)$ and determine when subalgebras of $\mathcal{E}(W)$ are in fact vertex algebras.

Recall the definition of a *weak vertex algebra*

Definition 3.1. A *weak vertex algebra* is a vector space V equipped with a linear map

$$\begin{aligned} Y(\cdot, x) :& V \to (End\ V)[[x, x^{-1}]] \\ & v \mapsto Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1} \end{aligned}$$

and a distinguished element $\mathbf{1} \in V$ satisfying the vacuum property and the creation property

$$Y(\mathbf{1}, x) = 1;$$
$$Y(v, x)\mathbf{1} \in V[[x]] \quad \text{and} \quad \lim_{x \to 0} Y(v, x)\mathbf{1} = v$$

for $v \in V$, and such that all the $\mathcal{D}$-bracket and derivative properties hold:

$$[\mathcal{D}, Y(v, x)] = Y(\mathcal{D}v, x) = \frac{d}{dx} Y(v, x)$$

where $\mathcal{D} \in End(V)$ is defined by

$$\mathcal{D}(v) = v_{-2}\mathbf{1}$$

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and most important, the definition of operations from the last time

$$Y_{\mathcal{E}}(a(x), x_0)b(x) = \mathrm{Res}_{x_1}\left(x_0^{-1}\delta\left(\frac{x_1 - x}{x_0}\right)a(x_1)b(x) - x_0^{-1}\delta\left(\frac{x - x_1}{-x_0}\right)b(x)a(x_1)\right)$$

and

$$a(x)_n b(x) = \mathrm{Res}_{x_1}((x_1 - x)^n a(x_1)b(x) - (-x + x_1)^n b(x)a(x_1))$$

Important case from the last talk was the case where a vertex algebra V is a substructure of $\mathcal{E}(W)$, a vertex algebra in which elements are weak vertex operators on W. Module structure of W has a nice form, where

$$Y_W(a(x), x_0) = a(x_0)$$

4. Locality

The key in determining vertex subalgebras of a canonical weak vertex algebra is the weak commutativity condition for pair of elements of $\mathcal{E}(W)$ acting on W. For $a(x), b(x) \in \mathcal{E}(W)$, we say that $a(x)$ and $b(x)$ are mutually local if there exists a nonnegative integer k such that

$$(x_1 - x_2)^k a(x_1)b(x_2) = (x_1 - x_2)^k b(x_2)a(x_1)$$

An easy way to think why the locality is important is that locality allows truncated conditions on mutually local elements. The general method to construct an example of a vertex algebra or a vertex operator algebra is as follows: we start with a collection of mutually local weak vertex operators on W, then we generate the weak vertex subalgebras of operators on W from them. The result will be a vertex algebra of "operators" on W and W will be a module for this algebra.

Definition 4.1. A weak vertex operator $a(x)$ in $\mathcal{E}(W)$ or $\mathcal{E}(W, d)$ is a *vertex operator* if it is local with itself, that is, there exists a nonnegative integer k such that

$$(x_1 - x_2)^k [a(x_1), a(x_2)] = 0$$

And we define a similar notion for a subspace

Definition 4.2. A subspace A of $\mathcal{E}(W)$ is *local* if for any $a(x), b(x) \in A$, $a(x)$ and $b(x)$ are mutually local. A *local subalgebra* of $\mathcal{E}(W)$ is a weak vertex subalgebra which is a local subspace.

Clearly, a local subspace consists of vertex operators. In case of W being a module for Virasoro, a notion of *graded local subspace* is defined in the obvious ways.

In this section, we will work toward proving that vertex subalgebras of $\mathcal{E}(W)$ are exactly the local subalgebras.

Theorem 4.3. Let V be a vertex subalgebra of $\mathcal{E}(W)$. Then V is a local subalgebra. Moreover, W is a module for V as a vertex algebra, with

$$Y_W(a(x), x_0) = a(x_0)$$

In particular, W is faithful

Proof. By the way $Y_{\mathcal{E}}(a(x), x_0)b(x)$ satisfies the iterate formula,

$$Y_{\mathcal{E}}(a(x), x_0)b(x) = \mathrm{Res}_{x_1}\left(x_0^{-1}\delta\left(\frac{x_1 - x}{x_0}\right)a(x_1)b(x) - x_0^{-1}\delta\left(\frac{x - x_1}{-x_0}\right)b(x)a(x_1)\right)$$

it follows that W is a V-module with $Y_W(a(x), x_0) = a(x_0)$. (from last week and Corollary 4.4.7). Since we have weak commutativity for Y_W, it follows that V is local. (Y_W is clearly injective) □

For the other direction, we start by proving the lower truncation condition.

Theorem 4.4. Let $a(x), b(x) \in \mathcal{E}(W)$ be mutually local with some $k \geq 0$ on W. Then $a(x)_n b(x) = 0$ for $n \geq k$.

Proof. From the previous week, we showed that each $a(x)_n$ maps $\mathcal{E}(W)$ into itself, and moreover that

$$a(x)_n b(x) = \mathrm{Res}_{x_1}((x_1 - x)^n a(x_1) b(x) - (-x + x_1)^n b(x) a(x_1))$$

Now the locality proves the assertion immediately.

□

The next is weak commutativity.

Theorem 4.5. Let $a(x), b(x) \in \mathcal{E}(W)$ be mutually local with $k \geq 0$ on W. Then

$$(x_1 - x_2)^k Y_{\mathcal{E}}(a(x), x_1) Y_{\mathcal{E}}(b(x), x_2) = (x_1 - x_2)^k Y_{\mathcal{E}}(b(x), x_2) Y_{\mathcal{E}}(a(x), x_1)$$

Proof. We prove the assertion by viewing them as operators. Let $c(x) \in \mathcal{E}(W)$. By definition,

$$\begin{aligned}
&Y_{\mathcal{E}}(a(x), x_1)(Y_{\mathcal{E}}(b(x), x_2) c(x)) \\
&= \mathrm{Res}_{x_3}\left(x_1^{-1}\delta\left(\frac{x_3 - x}{x_1}\right) a(x_3)(Y_{\mathcal{E}}(b(x), x_2)c(x)) - x_1^{-1}\delta\left(\frac{x - x_3}{-x_1}\right)(Y_{\mathcal{E}}(b(x), x_2)c(x)) a(x_3)\right) \\
&=^{(definition\ again)} \mathrm{Res}_{x_3}\mathrm{Res}_{x_4} x_1^{-1}\delta\left(\frac{x_3 - x}{x_1}\right) x_2^{-1}\delta\left(\frac{x_4 - x}{x_2}\right) a(x_3) b(x_4) c(x) \\
&- \mathrm{Res}_{x_3}\mathrm{Res}_{x_4} x_1^{-1}\delta\left(\frac{x_3 - x}{x_1}\right) x_2^{-1}\delta\left(\frac{x_4 - x}{x_2}\right) a(x_3) c(x) b(x_4) \\
&- \mathrm{Res}_{x_3}\mathrm{Res}_{x_4} x_1^{-1}\delta\left(\frac{x - x_3}{-x_1}\right) x_2^{-1}\delta\left(\frac{x_4 - x}{x_2}\right) b(x_4) c(x) a(x_3) \\
&+ \mathrm{Res}_{x_3}\mathrm{Res}_{x_4} x_1^{-1}\delta\left(\frac{x - x_3}{-x_1}\right) x_2^{-1}\delta\left(\frac{x_4 - x}{x_2}\right) c(x) b(x_4) a(x_3)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&Y_{\mathcal{E}}(b(x), x_2)(Y_{\mathcal{E}}(a(x), x_1) c(x)) \\
&= \mathrm{Res}_{x_3}\mathrm{Res}_{x_4} x_1^{-1}\delta\left(\frac{x_3 - x}{x_1}\right) x_2^{-1}\delta\left(\frac{x_4 - x}{x_2}\right) b(x_4) a(x_3) c(x) \\
&- \mathrm{Res}_{x_3}\mathrm{Res}_{x_4} x_1^{-1}\delta\left(\frac{x_3 - x}{x_1}\right) x_2^{-1}\delta\left(\frac{x_4 - x}{x_2}\right) b(x_4) c(x) a(x_3) \\
&- \mathrm{Res}_{x_3}\mathrm{Res}_{x_4} x_1^{-1}\delta\left(\frac{x - x_3}{-x_1}\right) x_2^{-1}\delta\left(\frac{x_4 - x}{x_2}\right) a(x_3) c(x) b(x_4) \\
&+ \mathrm{Res}_{x_3}\mathrm{Res}_{x_4} x_1^{-1}\delta\left(\frac{x - x_3}{-x_1}\right) x_2^{-1}\delta\left(\frac{x_4 - x}{x_2}\right) c(x) a(x_3) b(x_4)
\end{aligned}$$

Notice that the middle two lines are the same on both sides, so it is left to check equality on the first and the fourth lines. By (2.2.56) and proposition 2.3.26, we get

$$(x_1 - x_2)^k x_1^{-1}\delta\left(\frac{x_3 - x}{x_1}\right) x_2^{-1}\delta\left(\frac{x_4 - x}{x_2}\right) = (x_3 - x_4)^k x_1^{-1}\delta\left(\frac{x_3 - x}{x_1}\right) x_2^{-1}\delta\left(\frac{x_4 - x}{x_2}\right)$$

(in some sense, we are formally setting $(x_3 - x)/x_1 = 1$ and $(x_4 - x)/x_2 = 1$)

Now the assertion follows by multiplying $(x_1 - x_2)^k$.

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Now as in the previous week's conclusion, we have

Theorem 4.6. Local subalgebras of $\mathcal{E}(W)$ are preciely the vertex subalgebras, and W is a faithful module satisfying $Y_W(a(x), x_0) = a(x_0)$

We know how to determine vertex subalgebras now and the main tool is to start with a local subalgebra. Now, when do local subspaces generate local subalgebras?

Theorem 4.7. Let $a(x), b(x)$, and $c(x)$ be pairwise mutually local weak vertex operators on W. Then $a(x)_n b(x)$ and $c(x)$ are mutually local for all $n \in \mathbb{Z}$.

Proof. $a(x)_n b(x)$ is a weak vertex operator since $a(x)_n$ maps $\mathcal{E}(W)$ to $\mathcal{E}(W)$.

Let $n \in \mathbb{Z}$. Let r be a nonnegative integer $\geq -n$ such that

$$(x_1 - x_2)^r a(x_1)b(x_2) = (x_1 - x_2)^r b(x_2)a(x_1)$$
$$(x_1 - x_2)^r a(x_1)c(x_2) = (x_1 - x_2)^r c(x_2)a(x_1)$$
$$(x_1 - x_2)^r b(x_1)c(x_2) = (x_1 - x_2)^r c(x_2)b(x_1)$$

By definition,

$$a(x)_n b(x) = \mathrm{Res}_{x_1}((x_1 - x)^n a(x_1)b(x) - (-x + x_1)^n b(x)a(x_1))$$

Writing $(x - x_2)^{4r} = ((x - x_1) + (x_1 - x_2))^{3r}(x - x_2)^r$, we get

$$(x - x_2)^{4r}\big((x_1 - x)^n a(x_1)b(x)c(x_2) - (-x + x_1)^n b(x)a(x_1)c(x_2)\big)$$
$$= \sum_{k=0}^{3r} \binom{3r}{k} (x - x_1)^{3r-k}(x_1 - x_2)^k (x - x_2)^r \big((x_1 - x)^n a(x_1)b(x)c(x_2) - (-x + x_1)^n b(x)a(x_1)c(x_2)\big)$$

$((x_1 - x)^{r+(r+n)}[a(x_1), b(x)] = 0$ from the locality)

$$= \sum_{k=r+1}^{3r} \binom{3r}{k} (x - x_1)^{3r-k}(x_1 - x_2)^k (x - x_2)^r \big((x_1 - x)^n a(x_1)b(x)c(x_2) - (-x + x_1)^n b(x)a(x_1)c(x_2)\big)$$
$$= \sum_{k=r+1}^{3r} \binom{3r}{k} (x - x_1)^{3r-k}(x_1 - x_2)^k (x - x_2)^r \big((x_1 - x)^n c(x_2)a(x_1)b(x) - (-x + x_1)^n c(x_2)b(x)a(x_1)\big)$$
$$= \sum_{k=0}^{3r} \binom{3r}{k} (x - x_1)^{3r-k}(x_1 - x_2)^k (x - x_2)^r \big((x_1 - x)^n c(x_2)a(x_1)b(x) - (-x + x_1)^n c(x_2)b(x)a(x_1)\big)$$
$$= (x - x_2)^{4r}\big((x_1 - x)^n c(x_2)a(x_1)b(x) - (-x + x_1)^n c(x_2)b(x)a(x_1)\big)$$

Taking Res_{x_1}

$$(x - x_2)^{4r}\big(a(x)_n b(x)\big)c(x_2) = (x - x_2)^{4r} c(x_2)\big(a(x)_n b(x)\big)$$

$\square$

Theorem 4.8. Any maximal local subspace is a vertex algebra with W as a faithful module.

Proof. Let A be a maximal local subspace. 1_W is local with any vertex operator on W, so $1_W \in A$, other wise $A \subset A + \mathbb{C}1_W$.

Le $a(x), b(x) \in A$. From the above theorem, $a(x)_n b(x)$ is local with $a(x)$ and $b(x)$. Then again, $a(x)_n b(x)$ is local with itself. So $A + \mathbb{C}a(x)_n b(x)$ is again a local subspace and by the maximality, $A = A + \mathbb{C}a(x)_n b(x)$, showing that $Y_{\mathcal{E}}$ map is indeed over $End(A)$, thus A is a vertex algebra.

$\square$

Theorem 4.9. Let S be a local subset of $\mathcal{E}(W)$. Then S can be embedded in a vertex subalgebra of $\mathcal{E}(W)$ and the weak vertex algebra $\langle S\rangle$ is a vertex algebra. Furthermore, $\langle S\rangle = \operatorname{span}\{a^{(1)}(x)_{n_1}\cdots a(x)^{(r)}(x)_{n_r} : r \geq 0, a^{(i)}(x) \in S, n_i \in \mathbb{Z}\}$

Proof. Zorn's lemma guarantees a maximal local subspace V containing S, which in fact is a vertex subalgebra. Since $\langle S\rangle$ is a weak vertex algebra embedded in V, it is a vertex algebra.

It is clear that S is contained in that linear span by the creation property and any vertex subalgebra containing S contains that linear span. And follows from 3.9.3

□

5. "Operator"

Now we recall the subspace $\mathcal{E}(W,d)$ of $\mathcal{E}(W)$, the space of weak vertex operators on the pair (W,d), where d is a linear operator on W.

Definition 5.1. $\mathcal{E}(W,d) = \{a(x) \in \mathcal{E}(W) \mid [d, a(x)] = a'(x)\}$

Theorem 5.2. $\mathcal{E}(W,d)$ is a weak vertex subalgebra of $\mathcal{E}(W)$.

Proof. Since $\mathbf{1} \in \mathcal{E}(W,d)$, we must show that $\mathcal{D}$-bracket property holds for $a(x), b(x) \in \mathcal{E}(W,d)$

$$[d, Y_{\mathcal{E}}(a(x),x_0)b(x)] = \frac{\partial}{\partial x}Y_{\mathcal{E}}((a(x),x_0)b(x))$$

$$\begin{aligned}
&\frac{\partial}{\partial x}Y_{\mathcal{E}}((a(x),x_0)b(x)) \\
&= \frac{\partial}{\partial x}\operatorname{Res}_{x_1}\left(x_0^{-1}\delta\left(\frac{x_1-x}{x_0}\right)a(x_1)b(x) - x_0^{-1}\delta\left(\frac{x-x_1}{-x_0}\right)b(x)a(x_1)\right) \\
&= \operatorname{Res}_{x_1}\left(\left(\frac{\partial}{\partial x}x_0^{-1}\delta\left(\frac{x_1-x}{x_0}\right)\right)a(x_1)b(x) - \left(\frac{\partial}{\partial x}x_0^{-1}\delta\left(\frac{x-x_1}{-x_0}\right)\right)b(x)a(x_1)\right) \\
&+ \operatorname{Res}_{x_1}\left(x_0^{-1}\delta\left(\frac{x_1-x}{x_0}\right)a(x_1)b'(x) - x_0^{-1}\delta\left(\frac{x-x_1}{-x_0}\right)b'(x)a(x_1)\right) \\
&=^{(2.3.20)} -\operatorname{Res}_{x_1}\left(\left(\frac{\partial}{\partial x_1}x_0^{-1}\delta\left(\frac{x_1-x}{x_0}\right)\right)a(x_1)b(x) - \left(\frac{\partial}{\partial x}x_0^{-1}\delta\left(\frac{x-x_1}{-x_0}\right)\right)b(x)a(x_1)\right) \\
&+ \operatorname{Res}_{x_1}\left(x_0^{-1}\delta\left(\frac{x_1-x}{x_0}\right)a(x_1)b'(x) - x_0^{-1}\delta\left(\frac{x-x_1}{-x_0}\right)b'(x)a(x_1)\right) \\
&=^{(2.2.6)} \operatorname{Res}_{x_1}\left(x_0^{-1}\delta\left(\frac{x_1-x}{x_0}\right)a'(x_1)b(x) - x_0^{-1}\delta\left(\frac{x-x_1}{-x_0}\right)b(x)a'(x_1)\right) \\
&+ \operatorname{Res}_{x_1}\left(x_0^{-1}\delta\left(\frac{x_1-x}{x_0}\right)a(x_1)b'(x) - x_0^{-1}\delta\left(\frac{x-x_1}{-x_0}\right)b'(x)a(x_1)\right) \\
&= [d, Y_{\mathcal{E}}(a(x),x_0)b(x)]
\end{aligned}$$

□

Vertex operator algebras are equipped with grading and a structure associated with the Virasoro algebra. Let W be a module for the Virasoro algebra. As usual, let $L_W(x) = \sum_{n\in\mathbb{Z}} L(n)x^{-n-2}$, the conformal weight vector. W is a restricted module if for every $w \in W$, $L(n)w = 0$ for n sufficiently large. So $L_W(x)$ is a homogeneous weak vertex operator on $\mathcal{E}(W, L(-1))$ of weight two, since

$$[L_W(0), a(x)] = 2L_W(x) + xL'_W(x)$$

We will continue our discussion for $\mathcal{E}(W, L(-1))$ of a restricted module W for the Virasoro algebra.

Now we want to consider when $d = L(-1)$, where WW is a module for the Virasoro algebra. Recall the definition of the space $\mathcal{E}^o(W, L(-1))$, the space of homogeneous weak vertex operators on $(W, L(-1))$.

Definition 5.3. Let W be a module for the Virasoro algebra. A weak vertex operator $a(x)$ on W is *homogeneous of weight* $h \in \mathbb{C}$ if

$$[L(0), a(x)] = ha(x) + xa'(x)(= ha(x) + x[L(-1), a(x)] \quad \text{if } a(x) \text{ acts on } (W, L(-1)))$$

$\mathcal{E}^o(W, L(-1))$ is the linear span of all the homogeneous weak vertex operators on $(W, L(-1))$ such that

$$\mathcal{E}^o(W, L(-1)) = \coprod_{h \in \mathbb{C}} \mathcal{E}(W, L(-1))_{(h)}$$

where $\mathcal{E}(W, L(-1))_{(h)}$ is the space of homoegeneous weak vertex operators on $(W, L(-1))$ of weight h.

Our aim is to prove that any graded local subalgebra of $\mathcal{E}^o(W, L(-1))$ containing $L_W(x)$ is a vertex operator algebra with $L_W(x)$ playing the role of the conformal weight vector ω. (Without two grading restrictions $\dim V_(n) < \infty$ and $\dim V_(n) = 0$ for n large enough)

From the previous discussion, we have the following theorem

Theorem 5.4. Let W be a module for the Virasoro algebra. Let S be a local subset of homogeneous vertex operators on $(W, L(-1))$, such that the linear span of S is a graded local subspace of $\mathcal{E}^o(W, L(-1))$. Then $\langle S \rangle$ of $\mathcal{E}(W)$ is a graded vertex subalebra of $\mathcal{E}^o(W, L(-1))$ and $(W, L(-1))$ is a faithful module for $\langle S \rangle$ with $Y_W(a(x), x_0) = a(x_0)$. Moreover, $L(-1)$ bracket property holds in $\langle S \rangle$

$$Y_W(\mathcal{D}a(x), x_0) = [L(-1), Y_W(a(x), x_0)]$$

Suppose that W is a restricted module for the Virasoro algebra. Let S consist of a single element $L_W(x)$. Then $\langle L_W(x) \rangle$ is a $\mathbb{Z}$-graded vertex subalgebra of $\mathcal{E}^o(W, L(-1))$.

We would like to obtain vertex operator algebra structure from a given representation of the Virasoro algebra on W.

If V is a graded vertex subalgebra of $\mathcal{E}^o(W, L(-1))$ containing $L_W(x)$, then we would like to show that $L_W(x)$ is in fact a conformal vector of V.

First, we prove that $Y_{\mathcal{E}}(L_W(x), x_0)$ induces a representation of the Virasoro algebra of central charge l. Note that W is a faithful module for V.

We prove that if $\omega \in V$ is such that $Y_W(\omega, x)$ induces a representation of the Virasoro algebra, then $Y_{\mathcal{E}}(\omega, x)$ on V also induces a representation of the Virasoro algebra.

Theorem 5.5. Let U be a vector space, and let $h(x_1, x_2) \in U[[x_1, x_2, x_1^{-1}, x_2^{-1}]]$. If we can write $h(x_1, x_2) = \sum_{i=0}^{n} g_i(x_2)(\frac{\partial}{\partial x_1})^i x_2^{-1}\delta(x_1/x_2)$ such that $g_i(x_2) \in U[[x_2, x_2^{-1}]]$, then such expression is unique.

Theorem 5.6. Let V be a vertex algebra, let $u, v, w^{(0)}, \ldots, w^{(k)} \in V$ and let (W, Y_W) be a faithful V-module. Then

$$[Y(u, x_1), Y(v, x_2)] = \sum_{i=0}^{k} \frac{(-1)^i}{i!} Y(w^{(i)}, x_2)(\frac{\partial}{\partial x_1})^i x_2^{-1}\delta(\frac{x_1}{x_2})$$

if and only if

$$[Y_W(u,x_1),Y_W(v,x_2)]=\sum_{i=0}^{k}\frac{(-1)^i}{i!}Y_W(w^{(i)},x_2)(\frac{\partial}{\partial x_1})^i x_2^{-1}\delta(\frac{x_1}{x_2})$$

where $w^{(i)}=u_iv$ and $u_iv=0$ for all $i>k$.

Proof. By commutator formula,

$$\begin{aligned}[Y(u,x_1),Y(v,x_2)]&=\mathrm{Res}_{x_0}x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y(Y(u,x_0)v,x_2)\\&=\sum_{i=0}^{n}\frac{(-1)^i}{i!}Y(u_i,x_2)(\frac{\partial}{\partial x_1})^i x_2^{-1}\delta(\frac{x_1}{x_2})\end{aligned}$$

where n is large enough so that $u_iv=0$ for all $i>n$. This formula is from the Jacobi identity, so it indeed holds for Y_W as well. Now it follows from the previous theorem.

□

Theorem 5.7. $Y(\omega,x)$ induces a representation of the Virasoro algebra if and only if $Y_W(\omega,x)$ does.

Proof. The Virasoro algebra relations

$$[L_m,L_n]=(m-n)L(m+n)+\frac{l}{12}(m^3-m)\delta_{m+n,0}$$

in terms of generation functions is

$$\begin{aligned}[L_W(x_1),L_W(x_2)]&=\sum_{m,n\in\mathbb{Z}}(m-n)L(m+n)x_1^{-m-2}x_2^{-n-2}+\sum_{m\in\mathbb{Z}}\frac{l}{12}(m^3-m)x_1^{-m-2}x_2^{m-2}\\&=\sum_{m,n\in\mathbb{Z}}(-m-n-2)L(m+n)x_2^{-m-n-3}(x_1^{-m-2}x_2^{m+1})\\&+\sum_{m,n\in\mathbb{Z}}2(m+1)L(m+n)x_2^{-m-n-2}(x_1^{-m-2}x_2^{m})\\&+\sum_{m\in\mathbb{Z}}\frac{l}{12}(m^3-m)x_1^{-m-2}x_2^{m-2}\\&=L_W'(x_2)x_1^{-1}\delta(x_2/x_1)-2L_W(x_2)\frac{\partial}{\partial x_1}x_1^{-1}\delta(x_2/x_1)-\frac{l}{12}(\frac{\partial}{\partial x_1})^3x_1^{-1}\delta(x_2/x_1)\\&=L_W'(x_2)x_2^{-1}\delta(x_1/x_2)-2L_W(x_2)\frac{\partial}{\partial x_1}x_2^{-1}\delta(x_1/x_2)-\frac{l}{12}(\frac{\partial}{\partial x_1})^3x_2^{-1}\delta(x_1/x_2)\end{aligned}$$

giving us

$$\begin{aligned}&[Y(\omega,x_1),Y(\omega,x_2)]\\&=Y(\mathcal{D}\omega,x_2)x_2^{-1}\delta(x_1/x_2)-2Y(\omega,x_2)\frac{\partial}{\partial x_1}x_2^{-1}\delta(x_1/x_2)-\frac{l}{12}(\frac{\partial}{\partial x_1})^3x_2^{-1}\delta(x_1/x_2)\end{aligned}$$

Apply the above theorem with $k=3$, $\omega^{(0)}=\mathcal{D}\omega$, $\omega^{(1)}=2\omega$, $\omega^{(2)}=0$ and $\omega^{(3)}=(l/2)\mathbf{1}$.

□

Theorem 5.8. Let W be a restricted module for the Virasoro algebra with the cental charge l and let V be a graded vertex subalgebra of $\mathcal{E}^o(W, L(-1))$ containing $L_W(x)$. Then $(V, Y_{\mathcal{E}}, 1_W, L_W)$ is a vertex operator algebra with $\mathbb{C}$-grading and without two grading restrictions. Grading on $\mathcal{E}^o(W, L(-1))$ provides a grading on V by $L_V(0)$ eigenvalues.

Proof. Let $L_V(x_0) = \sum_{n\in\mathbb{Z}} L_V(n)x_0^{-n-2} = Y_{\mathcal{E}}(L_W(x), x_0)$, which induces a representation of the Virasoro algebra by the above disccussion on V.

Then by definition

$$\begin{aligned} L_V(-1)a(x) &= L_W(x)_0 a(x) = [L_W(-1), a(x)] = a'(x) = \mathcal{D}a(x) \\ L_V(0)a(x) &= L_W(x)_1 a(x) = [L_W(0), a(x)] - x[L_W(-1), a(x)] \\ \implies& [L_W(0), a(x)] = L_V(0)a(x) + x[L_W(-1), a(x)] \end{aligned}$$

Where the second line equals $ha(x)$ if $a(x) \in V_{(h)}$, proving the grading by eigenvalues. So $L(-1)$ bracket property holds on V, and

□

Now we may apply the theorem 4.9 in this setting.

Applying theorem 4.9 in the desired setting, we obtain $\mathbb{Z}$ grading and lots of properties of vertex operator algebras.

6. Construction Theorems

Theorem 5.5.18 of LL is in fact the foundation of the most construction theorems.

The following is the most general construction theorem

Theorem 6.1. Let V be a vector space equipped with a distinguished vector $\mathbf{1}$ and a linear operator d such that $d\mathbf{1} = 0$. Let T be a subset of V equipped with a "vertex" map

$$\begin{aligned} Y_0(\cdot, x) : &T \to Hom(V, V((x))) \\ &a \mapsto Y_0(a, x) = \sum_{n\in\mathbb{Z}} a_n x^{-n-1} \end{aligned}$$

such that for $a \in T$, vacuum and creation properties hold:

$$Y_0(a, x)\mathbf{1} \in V[[x]] \text{ and } \lim_{x\to 0} Y_0(a, x)\mathbf{1} = a$$

and the d-bracket property holds:

$$[d, Y_0(a, x)] = \frac{d}{dx} Y_0(a, x);$$

and for $a, b \in T$, we have locality (weak commutativity):

$$(x_1 - x_2)^k [Y_0(a, x_1), Y_0(b, x_2)] = 0;$$

and V is linearly spanned by the vectors "created by" T.

Then Y_0 can be uniquely extended to a linear map Y from V to $Hom(V, V((x)))$ such that $(V, Y, \mathbf{1})$ is a vertex algebra. Vertex algebra structure is obvious

$$Y(a^{(1)}_{n_1} \cdots a^{(r)}_{n_r}\mathbf{1}) = a^{(1)}(x)_{n_1} \cdots a^{(r)}(x)_{n_r} 1_V$$

where $a(x) = Y_0(a, x)$, and d on V satisfies

$$d(v) = v_{-2}\mathbf{1}$$

Furthermore, writing $T(x) = \{a(x) \mid a \in T\}$, we have an isomorphism of vertex algebras

$$\begin{aligned}\psi :& \langle T\rangle \to V \\ & \alpha(x) \mapsto \mathrm{Res}_x x^{-1}\alpha(x)\mathbf{1}\end{aligned}$$

Proof. First half follows easily from the theorems we have established already. So $\langle T\rangle$ is a vertex algebra with (V,d) as a module with $Y_V(\alpha(x), x_0) = \alpha(x_0)$. So the ψ map can be written as

$$\psi(\alpha(x)) = \mathrm{Res}_x x^{-1}\alpha(x)\mathbf{1} = \mathrm{Res}_{x_0} x_0^{-1} Y_V(\alpha(x), x_0)\mathbf{1}$$

where it follows from Chapter 4, that ψ is a $\langle T\rangle$-module isomorphism as (V,d) is a faithful module.

So V has a vertex algebra structure from such isomorphism. We determine key properties:

$$\psi(1_V) = \mathrm{Res}_{x_0} x_0^{-1} Y_V(1_V, x_0)\mathbf{1} = \mathrm{Res}_{x_0} x_0^{-1} 1_V(x_0)\mathbf{1} = 1_V\mathbf{1} = \mathbf{1}$$

showing that $\mathbf{1}$ is in fact the vacuum vector of V.

Using the property of homomorphism, Y map is defined naturally in that

$$Y(a, x_0) = \psi Y_{\mathcal{E}}(\psi^{-1}a, x_0)\psi^{-1} = Y_V(\psi^{-1}a, x_0) = Y_V(a(x), x_0) = a(x_0)$$

Formal differentiation operator $\mathcal{D}$ acting as $\mathcal{D}(v) = v_{-1}\mathbf{1}$ satisfies $[\mathcal{D}, Y(v,x)] = \frac{d}{dx}Y(v,x)$, and $\mathcal{D}$ and d act the same on $\mathbf{1}$, and V is spanned by things with $\mathbf{1}$, so d acts the same way as $\mathcal{D}$. $\square$

If we add in a condition that V is a restricted module for the Virasoro algebra, then we naturally get that V will be a vertex operator algebra.

7. Formal Calculus Identities

$$\mathrm{Res}_x(f'(x)v(x)) = -\mathrm{Res}_x(f(x)v'(x)) \tag{2.2.6}$$

$$\frac{\partial}{\partial x_1} x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) = -\frac{\partial}{\partial x_2} x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) = -\frac{\partial}{\partial x_0} x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) \tag{2.3.20}$$

$$x_1^{-1}\delta\left(\frac{x_2 - y}{x_1}\right) f(x_1, x_2, y) = x_1^{-1}\delta\left(\frac{x_2 - y}{x_1}\right) f(x_2 - y, x_2, y) \tag{2.3.56}$$