# NOTES FOR THE INTRODUCTORY PRESENTATION

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## 1. Introduction

A partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l)$ of a positive integer $n$ is a sequence of weakly descending positive integers such that $\sum_i \lambda_i = n$. For example, partitions of 5 are

$$(5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1), (1,1,1,1,1)$$

We denote the number of partitions of $n$ by $P(n)$, so $P(5) = 7$. From the above definition, we can write a generating function for $P(n)$:

$$\sum_{n\geq 0} P(n)q^n = \prod_{i>0} \frac{1}{1-q^i}$$
$$= (1 + q + q^{1+1} + ...)(1 + q^2 + q^{2+2} + ...)...(1 + q^n + q^{n+n} + ...)...$$

from which we see that each exponent gives us the multiplicity of $i$ in $\lambda$. (By convention $P(0) = 1$).

## 2. Examples

We can give some conditions to counting partitions. For example, the first Rogers-Ramanujan identity states that the number of partitions of $n$ whose parts are either congruent to 1 or $-1$ modulo 5 is equal to the number of partitions of $n$ such that $\lambda_i - \lambda_{i+1} \geq 2$ for all $i$. In terms of generating functions,

$$\prod_{i\geq 0} \frac{1}{(1-q^{1+5i})(1-q^{4+5i})} = \sum_{n\geq 0} \frac{q^{n^2}}{(1-q)(1-q^2)...(1-q^n)}$$

As seen above, when we speak of partition identities, we refer to the left side as the "sum side" and the right side as the "product side". Product sides usually have congruence conditions and sum sides have certain initial conditions and difference conditions along with some other conditions.

In fact, Gordon's identity generalizes Rogers-Ramanujan identities:

Let $B_{k,i}(n)$ the number of partitions of $n$ of the form $(b_1, b_2, ..., b_s)$ such that $b_j - b_{j+k-1} \geq 2$ and at most $i-1$ of the $b_j$ equal 1. Let $A_{k,i}(n)$ be the number of partitions of $n$ into parts $\not\equiv 0, \pm i \bmod 2k+1$. Then $A_{k,i}(n) = B_{k,i}(n)$ for all $n$.

Rogers-Ramanujan identity is a corollary of Gordon's generalization with $k = 2$ and $i = 2$.

Another family of partitions identities that looks like Rogers-Ramanujan-Gordon's is Goellnitz-Gordon-Andrews identities. The identity states:

Let $0 < i \leq k$ integers. Let $C_{k,i}(n)$ denote the number of partitions of $n$ into parts $\not\equiv 2 \bmod 4$ and $\not\equiv 0, \pm(2i-1) \bmod 4k$. Let $D_{k,i}(n)$ denote the number

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of partitions of the form $(d_1, d_2, ..., d_s)$ of $n$ such that no odd part is repeated, $d_j \geq d_{j+1}$, $d_j - d_{j+k-1} \geq 2$ if $d_j$ odd, $d_j - d_{j+k-1} > 2$ if $d_j$ even, and at most $i-1$ parts are $\leq 2$. Then $C_{k,i}(n) = D_{k,i}(n)$.

## 3. Identity Finder

In our project, we seek to develop new identities, new proofs of already existing partition identities and explore more algebraic aspects.

Our mentor Matthew Russell's `Maple` package `IdentityFinder` employs algorithms to develop new identities. We start with the sum side by counting the number of partitions under certain distance conditions to obtain a sequence of coefficients and convert it into the product side using the Euler's algorithm. If the result after Euler's algorithm seems to have some patterns, we conjecture a new identity.

## 4. Other projects

Other than finding new partitions, we also seek new "motivated proofs". "Motivated proofs" form a class of proofs that start with the product sides to deduce the sum sides. Recent work of our mentor Bud Coulson reinterpreted a motivated proof of Rogers-Ramanujan-Gordon identity with the affine Weyl group of $\widehat{\mathfrak{sl}(2)}$. We want to explore the relation and apply the idea to other identities.