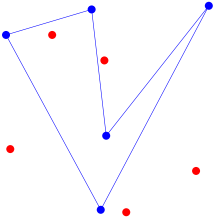
Empty Monochromatic Polygons in Unbalanced Random Point Sets

Presented by: Josef Cibulka
Joint work with: Martin Balko, Ondřej Bílka, Martin Böhm and Pavel Veselý
Mentor: James Abello

July 19, 2012 / DIMACS REU 2012

Points and polygons

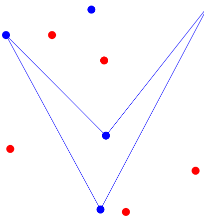
- Points in the plane, each is either blue or red
- *General position*: no three points in a line
- Polygons with vertices among the points
- *Simple polygons*: boundary does not cross itself
- Polygons do not need to be convex
- Example: blue pentagon



- A *hole* is a polygon with no other point inside

Empty monochromatic polygon

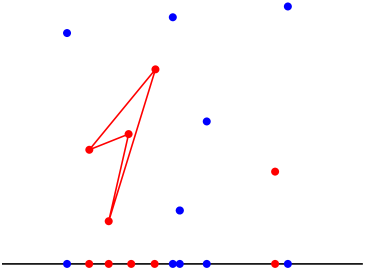
- Given k, find out if there is an n_0 such that every two-colored set of $n \geq n_0$ points has a monochromatic k-hole
- Example of a blue 4-hole



- Aichholzer et al. (2009): A set with $n \geq 5044$ points contains a monochromatic 4-hole.
- Remains open for $k \geq 5$.
- Examples of n-tuples of points avoiding monochromatic $\sqrt{n/2}$-holes.

Holes in random point sets

- Points drawn randomly from a unit square: $n/2$ blue points and $n/2$ red points
- *Size of a polygon:* number of vertices
- By a result of Balogh et al. (2012), the maximum size of a **convex** (monochromatic) hole is almost surely $\Theta(\log(n)/\log(\log(n)))$.
- A (not necessarily convex) monochromatic hole of size $\Omega(\log(n))$ exists almost surely: Project to the x-axis and find a consecutive sequence of $\log(n)/2$ red points.



Unbalanced point sets

Theorem

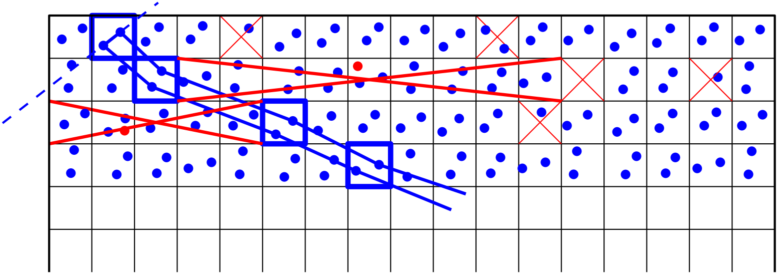
Given a set of $1000n$ *blue points and* $n/1000$ *red points (n is large enough), we almost surely find a blue hole of size* $\sqrt{n}/const$*.*

Unbalanced point sets - sketch of the proof

- Grid $\sqrt{n} \times \sqrt{n}$.
- Go from a cell in the left half of the top row always one row lower and 1 to 3 cells to the right until visiting $\sqrt{n}/20$ cells

Safe path on grid cells:

- Each cell contains two (or more) blue points whose connecting line has a positive slope.
- For each cell, the distance from the nearest red point in the same row is more than four cells.



Existence of a safe path: Reformulation

- Vertical direction is time.
- A random process on a row of m trees
- Place i contains a tree after step $t \Leftrightarrow$ cell $(t, i + 2t)$ is safely reachable
- Each step:
  - rebuilding phase: tree reappears on a place at distance at most one from a tree
  - destructive phase: randomly burn sequences of nine trees
- Is it almost sure that some tree exists after m steps?