Concordance Invariants of Satellite Knots

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A knot is a loop of string in $\mathbb{R}^3$, which has no thickness, with its cross-section being a single point. (Formally, we say a knot is an embedding $S^1 \hookrightarrow S^3$.)

(a) The unknot. (b) A trefoil knot.

There are many different pictures of the same knot. Below are all pictures of the figure eight knot.
We study knots because they are closely related to 3 and 4 dimensional manifolds.

**Theorem (Lickorish, Wallace, 1960s)**

Every closed orientable 3-dimensional manifold can be described in terms of a collection of knots and an integer associated to each knot.

Note that 3D manifolds are hard to visualize, but knots are not!
Knots are often studied up to a notion of equivalence, called knot concordance.

Two knots are said to be **concordant** if they jointly form the boundary of a cylinder in $S^3 \times [0, 1]$.

Formally speaking, two knots $K$ and $J$ are said to be **concordant** ($K \sim J$) if there is an embedding $f : S^1 \times [0, 1] \to S^3 \times [0, 1]$ such that $f(S^1 \times 0) = K$ and $f(S^1 \times 1) = J$.

The set of concordance classes of knots form a group, denoted $\mathcal{C}$.
In 2003, P. Ozsváth and Z. Szabó defined an invariant of the concordance class of a knot, called the $\tau$-invariant. Formally, the $\tau$-invariant is a group homomorphism $\tau : \mathcal{C} \rightarrow \mathbb{Z}$ which sends all elements of a concordance class to an integer.

J. Hom defined the $\epsilon$-invariant, valued in $\{-1, 0, 1\}$.

The goal of this project is to compute $\tau$ and $\epsilon$ for specific types of knots (denoted by $P(K)$), called satellite knots.
A satellite knot has two components: a pattern knot $P$ (embedded in a solid torus) and a companion knot $K$. Cut up the torus and glue it back according to $K$. The image of $P$ under this process is called the satellite knot with pattern $P$ and companion $K$, denoted by $P(K)$.

E.g., let $P$ be the Whitehead double, $K$ be the figure eight.
We are interested in the satellite knots coming from the Mazur pattern $Q$, shown below, as well as generalizations of this pattern $Q_{m,n}$. 
In 2016, A. Levine used a family of knot invariants called **bordered knot Floer homology** to give a formula of the tau-invariant of satellite knots with Mazur patterns.

**Theorem (Levine, 2016)**

Let $Q$ denote the Mazur pattern. For any knot $K \subset S^3$,

$$
\tau(Q(K)) = \begin{cases} 
\tau(K) & \text{if } \tau(K) \leq 0 \text{ and } \epsilon(K) \in \{0, 1\}, \\
\tau(K) + 1 & \text{if } \tau(K) > 0 \text{ or } \epsilon(K) = -1.
\end{cases}
$$

Our first goal is to simulate this process to compute tau-invariant for general Mazur patterns $Q_{m,n}$. 

Main Theorem

For any knot $K \subset S^3$, we have

$$\tau(Q_{m,n}(K)) = \begin{cases} 
|m - n|\tau(K) + (m - 1) & \text{if } \tau(K) > 0 \text{ and } m > n, \\
|m - n|\tau(K) + m & \text{if } \tau(K) > 0 \text{ and } m \leq n, \\
(m - n)\tau(K) + (m - 1) & \text{if } \tau(K) \leq 0 \text{ and } \epsilon(K) = -1, \\
(m - n)\tau(K) & \text{if } \tau(K) \leq 0 \text{ and } \epsilon(K) = 0, 1. 
\end{cases}$$

In particular, when the winding number of $Q_{m,n}$ is 1:

$$\tau(Q_{m,n}(K)) = \begin{cases} 
\tau(K) + m & \text{if } \tau(K) > 0, \\
-\tau(K) + (m - 1) & \text{if } \tau(K) \leq 0 \text{ and } \epsilon(K) = -1, \\
-\tau(K) & \text{if } \tau(K) \leq 0 \text{ and } \epsilon(K) = 0, 1. 
\end{cases}$$

When the winding number is $-1$:

$$\tau(Q_{m,n}(K)) = \begin{cases} 
\tau(K) + (m - 1) & \text{if } \tau(K) > 0 \text{ or } \epsilon(K) = -1, \\
\tau(K) & \text{if } \tau(K) \leq 0 \text{ and } \epsilon(K) = 0, 1. 
\end{cases}$$
To a 3-manifold $Y$, we associate two invariants: \( CFD^*(Y) \) and \( CFA^*(Y) \).

The pairing theorem states that for a pattern knot $P \subset V = S^1 \times D^2$ and a companion knot $K$, we have

\[
gCFK^*(S^3, P(K)) \simeq CFA^*(V, P) \boxtimes CFD^*(X_K),
\]

Once we have \( gCFK^*(S^3, P(K)) \), calculating the $\tau$-invariant for $P(K)$ is easy.

We know \( CFD^*(X_K) \) from literature. We can calculate \( CFA^*(V, P) \) combinatorially via bordered Heegaard diagrams of $P$. 

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Bordered Heegaard Diagrams

To each knot $P \subset V$, we can associate a **bordered Heegaard diagram**

From the bordered Heegaard diagrams, we enumerate all the "pseudoholomorphic disks" to recover $\text{CFA}^*(V, P)$.
Strategy to Construct Bordered Diagrams for $Q_{m,n}$
Example: Calculating $Q_{1,2}$

The bordered Heegaard diagram for $Q_{1,2}$ is given by
Example: Calculating $Q_{1,2}$

The complex $CFA^*(V, Q_{1,2})$ is:

For any knot $K \subset S^3$, the complex $CFD^*(X_K)$ looks like

By the pairing theorem, we obtain the tensor complex

$$gCFK^*(P(K)) = CFA^*(V, Q_{1,2}) \otimes CFD^*(X_K):$$
The calculation for $\epsilon(Q_{m,n}(K))$ amounts to finding $\tau(Q_{m,n}(K)_{2,1})$ and $\tau(Q(K)_{2,-1})$.

We do this calculation via an algorithm designed by R. Lipshitz, P. Ozsváth and D. Thurston, and implemented in Python by B. Zhan.

**Question (Akbulut, 1997)**

Does there exist a winding number $\pm 1$ satellite operator $P$ for which $P(K)$ is never exotically slice?

- Levine’s paper answered this in the affirmative, with $P$ as the Mazur pattern.
- If $\epsilon(Q_{m,n}(K))$ turns out as expected when the winding number is $\pm 1$, we would have found a large family of examples that answer the aforementioned question.
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Thanks for your time!