Concordance Invariants of Satellite Knots

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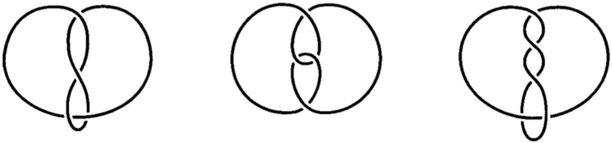
What is a knot?

A **knot** is a loop of string in $\mathbb{R}^3$, which has no thickness, with its cross-section being a single point. (Formally, we say a knot is an embedding $S^1 \hookrightarrow S^3$.)

(a) The unknot. (b) A trefoil knot.

There are many different pictures of the same knot. Below are all pictures of the figure eight knot.



Why knots?

We study knots because they are closely related to 3 and 4 dimensional manifolds.

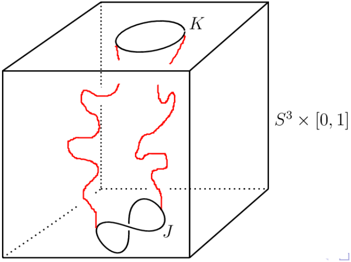
Theorem (Lickorish, Wallace, 1960s)

Every closed orientable 3-dimensional manifold can be described in terms of a collection of knots and an integer associated to each knot.

Note that 3D manifolds are hard to visualize, but knots are not!

Concordance

- Knots are often studied up to a notion of equivalence, called knot concordance.
- Two knots are said to be **concordant** if they jointly form the boundary of a cylinder in $S^3 \times [0, 1]$.
- Formally speaking, two knots K and J are said to be **concordant** ($K \sim J$) if there is an embedding $f : S^1 \times [0, 1] \to S^3 \times [0, 1]$ such that $f(S^1 \times 0) = K$ and $f(S^1 \times 1) = J$
- The set of concordance classes of knots form a group, denoted $\mathscr{C}$.

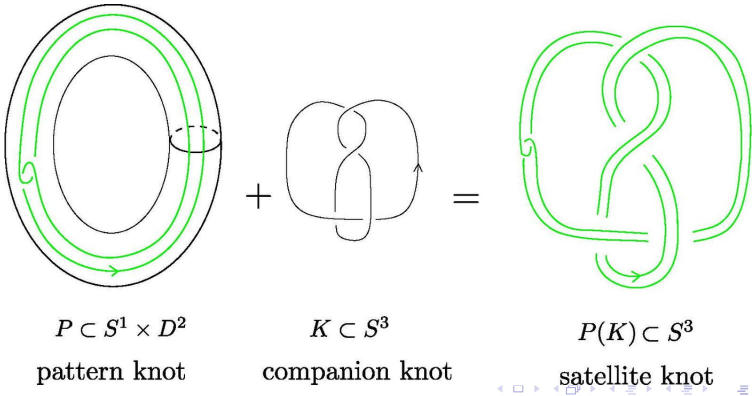


Two types of Concordance Invariants

- In 2003, P. Ozsváth and Z. Szabó defined an invariant of the concordance class of a knot, called the τ-invariant.
- Formally, the τ-invariant is a group homomorphism $\tau : \mathscr{C} \to \mathbb{Z}$ which sends all elements of a concordance class to an integer.
- J. Hom defined the ϵ-invariant, valued in $\{-1, 0, 1\}$.
- The goal of this project is to compute τ and ϵ for specific types of knots (denoted by $P(K)$), called satellite knots.

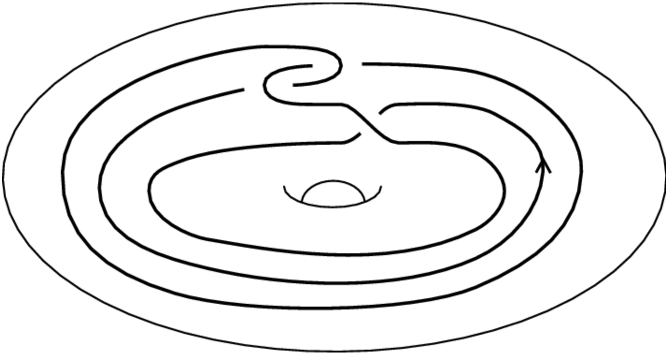
Satellite Knot

- A satellite knot has two components: a pattern knot P (embedded in a solid torus) and a companion knot K. Cut up the torus and glue it back according to K. The image of P under this process is called the **satellite knot with pattern P and companion K**, denoted by $P(K)$.
- E.g., let P be the Whitehead double, K be the figure eight



Mazur Pattern

We are interested in the satellite knots coming from the Mazur pattern Q, shown below, as well as generalizations of this pattern $Q_{m,n}$.



Bordered Knot Floer Homology

In 2016, A. Levine used a family of knot invariants called **bordered knot Floer homology** to give a formula of the tau-invariant of satellite knots with Mazur patterns.

Theorem (Levine, 2016)

Let Q denote the Mazur pattern. For any knot $K \subset S^3$,

$$\tau(Q(K)) = \begin{cases} \tau(K) & \text{if } \tau(K) \leq 0 \text{ and } \epsilon(K) \in \{0, 1\}, \\ \tau(K) + 1 & \text{if } \tau(K) > 0 \text{ or } \epsilon(K) = -1. \end{cases}$$

Our first goal is to simulate this process to compute tau-invariant for general Mazur patterns $Q_{m,n}$.

Main Theorem

For any knot $K \subset S^3$, we have

$$\tau(Q_{m,n}(K)) = \begin{cases} |m-n|\tau(K) + (m-1) & \text{if } \tau(K) > 0 \text{ and } m > n, \\ |m-n|\tau(K) + m & \text{if } \tau(K) > 0 \text{ and } m \leq n, \\ (m-n)\tau(K) + (m-1) & \text{if } \tau(K) \leq 0 \text{ and } \epsilon(K) = -1, \\ (m-n)\tau(K) & \text{if } \tau(K) \leq 0 \text{ and } \epsilon(K) = 0, 1. \end{cases}$$

In particular, when the winding number of $Q_{m,n}$ is 1:

$$\tau(Q_{m,n}(K)) = \begin{cases} \tau(K) + m & \text{if } \tau(K) > 0, \\ -\tau(K) + (m-1) & \text{if } \tau(K) \leq 0 \text{ and } \epsilon(K) = -1, \\ -\tau(K) & \text{if } \tau(K) \leq 0 \text{ and } \epsilon(K) = 0, 1. \end{cases}$$

When the winding number is -1:

$$\tau(Q_{m,n}(K)) = \begin{cases} \tau(K) + (m-1) & \text{if } \tau(K) > 0 \text{ or } \epsilon(K) = -1, \\ \tau(K) & \text{if } \tau(K) \leq 0 \text{ and } \epsilon(K) = 0, 1. \end{cases}$$

Strategy

- To a 3-manifold Y, we associate two invariants: $CFD^*(Y)$ and $CFA^*(Y)$.
- The pairing theorem states that for a pattern knot $P \subset V = S^1 \times D^2$ and a companion knot K, we have

$$gCFK^*(S^3, P(K)) \simeq CFA^*(V, P) \boxtimes CFD^*(X_K),$$

- Once we have $gCFK^*(S^3, P(K))$, calculating the τ-invariant for $P(K)$ is easy.
- We know $CFD^*(X_K)$ from literature. We can calculate $CFA^*(V, P)$ combinatorially via bordered Heegaard diagrams of P.

Bordered Heegaard Diagrams

To each knot $P \subset V$, we can associate a **bordered Heegaard diagram**

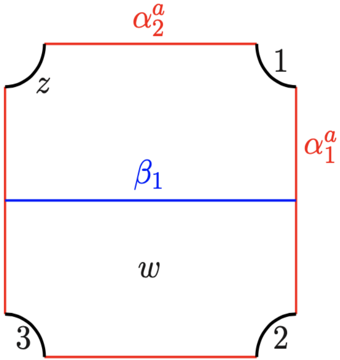


Diagram for the trivial pattern

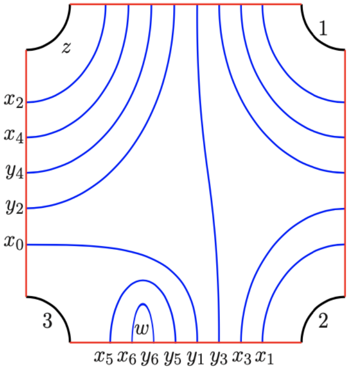
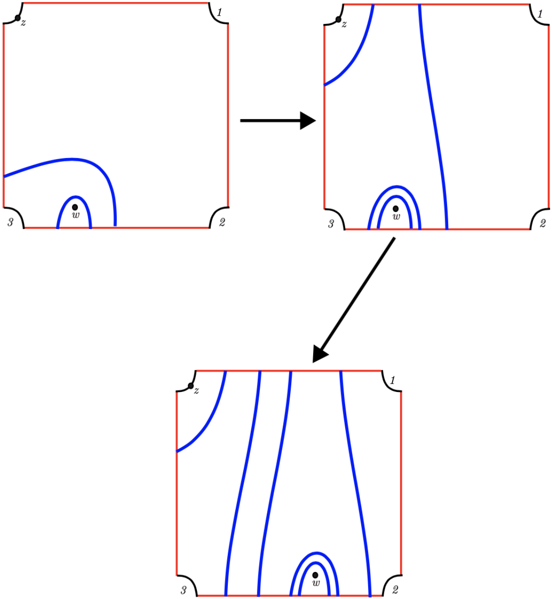


Diagram for the Mazur pattern

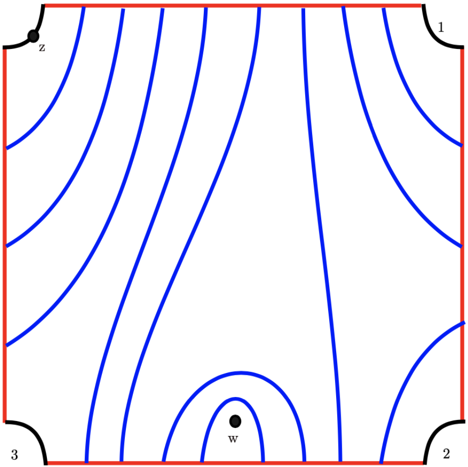
From the bordered Heegaard diagrams, we enumerate all the "pseudoholomorphic disks" to recover $CFA^*(V, P)$.

Strategy to Construct Bordered Diagrams for $Q_{m,n}$



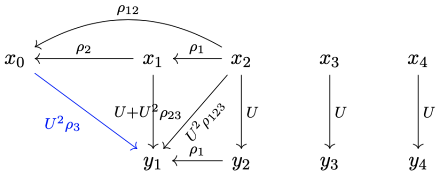
Example: Calculating $Q_{1,2}$

The bordered Heegaard diagram for $Q_{1,2}$ is given by



Example: Calculating $Q_{1,2}$

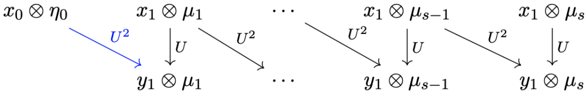
The complex $CFA^*(V, Q_{1,2})$ is:



For any knot $K \subset S^3$, the complex $CFD^*(X_K)$ looks like

$$\eta_0 \xrightarrow{D_3} \mu_1 \xrightarrow{D_{23}} \cdots \xrightarrow{D_{23}} \mu_s \xleftarrow{D_1} \xi_0$$

By the pairing theorem, we obtain the tensor complex $gCFK^*(P(K)) = CFA^*(V, Q_{1,2}) \boxtimes CFD^*(X_K)$:



Calculation for $\epsilon(Q_{m,n}(K))$

- The calculation for $\epsilon(Q_{m,n}(K))$ amounts to finding $\tau(Q_{m,n}(K)_{2,1})$ and $\tau(Q(K)_{2,-1})$.
- We do this calculation via an algorithm designed by R. Lipshitz, P. Ozsváth and D. Thurston, and implemented in Python by B. Zhan.

Question (Akbulut, 1997)

Does there exist a winding number ±1 satellite operator P for which $P(K)$ is never exotically slice?

- Levine's paper answered this in the affirmative, with P as the Mazur pattern.
- If $\epsilon(Q_{m,n}(K))$ turns out as expected when the winding number is ±1, we would have found a large family of examples that answer the aforementioned question.

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Thanks for your time!