When Fourier Analysis Meets Ergodic Theory and Combinatorics

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What is Ergodic Theory?

Ergodic theory allows for the study of the statistical properties of a **dynamical system**, or a probability space together with a 'measure-preserving' map T.

Birkhoff's Ergodic Theorem

Birkhoff's ergodic theorem establishes pointwise convergence for "time averages" to a "space averages" as follows:

Theorem (Birkhoff)

Let $(X, \mathcal{B}, \mu, T)$ be an ergodic measure preserving system, and $f \in L^1_\mu$. Then,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f \circ T^n(x) = \int_X f \, d\mu$$

μ-a.e. for $x \in X$.

The quantity on the left is the "time average," and the quantity on the right is the "space average."

Our goal

We aim to establish the following theorem:

Theorem

Let $(X, \mathcal{B}, \mu, T)$ be a σ-finite dynamical system with T invertible. Let $c > 1$ be sufficiently close to 1, and $(p_n)_{n\in\mathbb{N}}$ be the standard enumeration of primes. For all $r \in (1, \infty)$ and $f \in L^r_\mu$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f \circ T^{\lfloor p_n^c \rfloor}(x) \text{ exists for } \mu\text{-a.e. } x \in X,$$

The plan

- Our problem can be reduced to the "universal" ergodic m.p.s. $(\mathbb{Z}, \mathcal{P}(\mathbb{Z}), \mu, T)$ where μ is the counting measure and $T(x) = x - 1$ (part of our project is understanding this)
- We will study the problem using concepts from harmonic analysis (such as Fourier series)
- We will use techniques from analytic number theory to understand the orbits of the sequence p_n^c

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